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Group classification of equations of mathematical physics

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VII Conference on Modern Mathematical Physics
Belgrade, Serbia
September 9–19, 2012

Introduction

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- "A physical law must possess mathematical beauty"
- "It is more important for our equations to be beautiful than to have them fit experiment"

Symmetry

can be considered as a measure of beauty

for partial differential equations

Introduction

- A *symmetry* of a PDE system is any transformation that maps a solution of this system into a solution of the same system.
- A particular important class of symmetries is Lie symmetries, which correspond to Lie groups of continuous point transformations.

Introduction

All the basic equations of mathematical physics, i.e. the equations of Newton, Laplace, d'Alembert, Euler-Lagrange, Lame, Hamilton-Jacobi, Maxwell, Schrödinger etc., have nontrivial symmetry properties.

It means that manifolds of their solutions are invariant with respect to multi-parameter group of continuous transformations (Lie group of transformations) with large number of parameters.

This property distinguishes these equations from other partial differential equations considered by mathematicians.

Introduction

The problem: To single out equations having high symmetry properties from a given class of PDEs.

Given a class of differential equations, to classify all possible cases of extension of Lie invariance algebras of such equations with respect to the equivalence group of the class.

A comprehensive investigation of this problem involves:

- Finding the kernel of the maximal Lie invariance algebras
- Constructing the equivalence group of the entire class
- Describing all possible inequivalent equations from the class that admit maximal Lie invariance algebras properly containing the kernel.

Group classification problem

Two main approaches of solving group classification problems

1. Based on subgroup analysis of the equivalence group of a class of differential equations under consideration.
   (It can be applied if the class is normalized)
2. Involves investigation of compatibility and the direct integration of determining equations implied by the infinitesimal invariance criterion
   (It is efficient only for classes of a simple structure, having a few arbitrary elements)

Group classification problem

To solve more group classification problems and to present results in an optimal way, different tools and notions were recently proposed

- extended and generalized equivalence groups
- conditional equivalence group
- gauging of arbitrary elements by equivalence transformations
- partition of a class to normalized subclasses
- mappings between classes of PDEs

Variable coefficient nonlinear reaction–diffusion equations

$$\mathbf{f(x)u_t = (g(x)u^n u_x)_x + h(x)u^m, \quad n \neq 0} \qquad (1)$$

We search for operators of the form $\mathbf{Q = \tau(t,x,u)\partial_t + \xi(t,x,u)\partial_x + \eta(t,x,u)\partial_u}$, which generate one-parameter groups of point symmetry transformations of equations from class (1). These operators satisfy the necessary and sufficient criterion of infinitesimal invariance, i.e. we require that

$$\mathbf{Q^{(2)}\left(fu_t - gu^n u_{xx} - ngu^{n-1}u_x^2 - g_x u^n u_{xx} - hu^m\right) = 0}$$

identically, modulo equation (1).

Determining equations

Some of the determining equations does not contain arbitrary elements and therefore can be integrated immediately. Others (i.e. the equations containing arbitrary elements explicitly) are called classifying equations. The main difficulty of group classification is the need to solve classifying equations with respect to the coefficients of the operator Q and arbitrary elements simultaneously.

Theorem

The usual equivalence group $G^{\sim}$ of the class

$$\mathbf{f(x)u_t = (g(x)u^n u_x)_x + h(x)u^m}$$

consists of the transformations

$$\tilde{t} = \delta_1 t + \delta_2, \quad \tilde{x} = \varphi(x), \quad \tilde{u} = \delta_3 u,$$

$$\tilde{f} = \frac{\delta_0\delta_1}{\delta_3\varphi_x} f, \quad \tilde{g} = \frac{\delta_0\varphi_x}{\delta_3^{n+1}} g, \quad \tilde{h} = \frac{\delta_0}{\delta_3^m\varphi_x} h, \quad \tilde{n} = n, \quad \tilde{m} = m,$$

where δ_j, $j = 0, \dots, 3$ are arbitrary constants, $\delta_0\delta_1\delta_3 \neq 0$, φ is an arbitrary smooth function of x, $\varphi_x \neq 0$.

Conditional equivalence group

Sometimes it happens that equivalence group of certain subclass of the class under study is wider than equivalence group of the entire class. Such equivalence groups are called **conditional** ones.

Theorem

The class of equations $\mathbf{f(x)u_t = (g(x)u^n u_x)_x + h(x)u^{n+1}}$ *admits the equivalence group* $G^{\sim}_{m=n+1}$ *consisting of the transformations:*

$$\tilde{t} = \delta_1 t + \delta_2, \quad \tilde{x} = \varphi(x), \quad \tilde{u} = \psi(x)u,$$

$$\tilde{f} = \frac{\delta_0\delta_1}{\psi^{n+2}\varphi_x} f, \quad \tilde{g} = \frac{\delta_0\varphi_x}{\psi^{2n+2}} g, \quad \tilde{h} = \delta_0 \frac{h - \psi^{n+1}(\psi^{-(n+2)}\psi_x g)_x}{\psi^{2n+2}\varphi_x},$$

where φ *and* ψ *are arbitrary functions of* x, δ_j $(j = 0, 1, 2)$ *are arbitrary constants,* $\delta_0\delta_1\varphi_x\psi \neq 0$.

Conditional equivalence group

Other subclass admitting nontrivial conditional group is

$$\mathbf{f(x)u_t = (g(x)u^n u_x)_x + \varepsilon f(x)\,u}$$

Results of group classification: General case

	$f(x)$	$h(x)$	Basis of A^{max}
1	$\forall$	$\forall$	∂_t
2	$f_1(x)$	$h_1(x)$	$\partial_t, (d+2b-pn)t\partial_t + ((n+1)ax^2+bx+c)\partial x + (ax+p)u\partial_u$
3	1	ε	$\partial_t, \partial_x, 2(1-m)t\partial_t + (1+n-m)x\partial_x + 2u\partial_u$

Here $\varepsilon = \pm 1$, n is arbitrary for all cases.

$$f_1(x) = e^{\int \frac{-(3n+4)ax+d}{(n+1)ax^2+bx+c}dx}, \quad h_1(x) = e^{\int \frac{-(3(n+1)+m)ax+(1+m-n)p-2b}{(n+1)ax^2+bx+c}dx},$$

and it can be assumed up to equivalence with respect to $G_1^{\sim}$ that the parameter tuple (a, b, c, d) takes only the following non-equivalent values:

$$(\varepsilon, 0, 1, 0) \quad \text{if} \quad n = -1 \quad \text{or} \quad (1, 0, 1, d') \quad \text{if} \quad n \neq -1, \quad (0, 1, 0, d'), \quad (0, 0, 1, 1),$$

where d' is arbitrary constant. In all the cases we put $g(x) = 1$.

Results of group classification: $m = 1,\ h \neq 0,\ (h/f)_x = 0$

	n	$f(x)$	$h(x)$	Basis of A^{max}
4	$\forall$	$\forall$	εf	$\partial_t, \quad e^{-\varepsilon nt}(\partial_t + \varepsilon u\partial_u)$
5	$\forall$	$f_1(x)$	εf	$\partial_t, \quad e^{-\varepsilon nt}(\partial_t + \varepsilon u\partial_u),$ $n((n+1)ax^2 + bx + c)\partial x + (nax + 2b + d)u\partial_u$
6	$\neq -\frac{4}{3}$	1	ε	$\partial_t, \quad \partial_x, \quad e^{-\varepsilon nt}(\partial_t + \varepsilon u\partial_u), \quad nx\partial x + 2u\partial_u$
7	$-\frac{4}{3}$	1	ε	$\partial_t, \quad \partial_x, \quad e^{\frac{4}{3}\varepsilon t}(\partial_t + \varepsilon u\partial_u),$ $-\frac{4}{3}x\partial x + 2u\partial_u, \quad -\frac{1}{3}x^2\partial_x + xu\partial_u$

$$f_1(x) = \exp\left[\int \frac{-(3n+4)ax + d}{(n+1)ax^2 + bx + c}dx\right],$$

and it can be assumed up to equivalence with respect to $G_1^{\sim}$ that the parameter tuple (a, b, c, d) takes only the following non-equivalent values:

$$(\varepsilon, 0, 1, 0) \quad \text{if} \quad n = -1 \quad \text{or} \quad (1, 0, 1, d') \quad \text{if} \quad n \neq -1, \quad (0, 1, 0, d'), \quad (0, 0, 1, 1),$$

where d' is arbitrary constant. In all the cases we put $g(x) = 1$.

Results of group classification: $m = n + 1$ or $h = 0$

	n	$f(x)$	$h(x)$	Basis of A^{max}
8	$\forall$	$\forall$	$\forall$	$\partial_t, nt\partial_t - u\partial_u$
9	$\neq -\frac{4}{3}$	1	αx^{-2}	$\partial_t, nt\partial_t - u\partial_u, 2t\partial_t + x\partial_x$
10	$\neq -\frac{4}{3}$	1	ε	$\partial_t, nt\partial_t - u\partial_u, \partial_x$
11	$\neq -\frac{4}{3}$	1	0	$\partial_t, \partial_x, nt\partial_t - u\partial_u, 2t\partial_t + x\partial_x$
12	$-\frac{4}{3}$	e^x	α	$\partial_t, t\partial_t + \frac{3}{4}u\partial_u, \partial_x - \frac{3}{4}u\partial_u$
13	$-\frac{4}{3}$	1	0	$\partial_t, \partial_x, \frac{4}{3}t\partial t + u\partial_u, 2t\partial_t + x\partial_x, -\frac{1}{3}x^2\partial_x + xu\partial_u$

Here α is arbitrary constant, $\alpha \neq 0$ in case 9, $\varepsilon = \pm 1$.
In case 8 the parameter-functions f and h can be additionally gauged with equivalence transformations from $G^{\sim}_{1,m=n+1}$. For example, we can put $f = 1$ if $n \neq -4/3$ and $f = e^x$ otherwise.

KdV- and mKdV-like equations

The general form of the variable-coefficient KdV-like equations is

$$u_t + f(t)uu_x + g(t)u_{xxx} + h(t)u + (p(t) + q(t)x)u_x + k(t)x + l(t) = 0.$$

The variable-coefficient mKdV-like equations have the general form

$$u_t + f(t)u^2u_x + g(t)u_{xxx} + h(t)u + (p(t) + q(t)x)u_x + k(t)uu_x + l(t) = 0.$$

Example of reducibility to the mKdV equation

In several recent papers the authors studied “the mKdV equations with variable coefficients” having the form

$$\mathbf{u_t = K_0(t)(u_{xxx} - 6u^2u_x) + 4K_1(t)u_x - h(t)(u + xu_x)},$$

where K_0, K_1 and h are arbitrary (smooth) functions of t, $K_0 \neq 0$. Some exact solutions were found and it was proved that this equation is integrable.

Example of reducibility to the mKdV equation

Any equation of the form

$$\mathbf{u_t = K_0(t)(u_{xxx} - 6u^2u_x) + 4K_1(t)u_x - h(t)(u + xu_x),} \quad (2)$$

is equivalent to the standard mKdV equation
$\tilde{\mathbf{u}}_{\tilde{\mathbf{t}}} - 6\tilde{\mathbf{u}}^2\tilde{\mathbf{u}}_{\tilde{\mathbf{x}}} + \tilde{\mathbf{u}}_{\tilde{\mathbf{x}}\tilde{\mathbf{x}}\tilde{\mathbf{x}}} = \mathbf{0}$ with respect to the point transformation

$$\tilde{\mathbf{t}} = \alpha(\mathbf{t}), \quad \tilde{\mathbf{x}} = \beta(\mathbf{t})\mathbf{x} + \gamma(\mathbf{t}), \quad \tilde{\mathbf{u}} = \frac{\mathbf{u}}{\beta(\mathbf{t})},$$

where the functions α, β and γ have the form

$$\alpha = -\int \mathbf{K_0 e^{-3\int h\,dt} dt}, \ \beta = \mathbf{e^{-\int h\,dt}}, \ \gamma = 4\int \mathbf{K_1 e^{-\int h\,dt} dt}.$$

The function $u = u(t, x)$ satisfies equation (2) if and only if there exists a solution $\tilde{u} = \tilde{u}(\tilde{t}, \tilde{x})$ of the standard mKdV equation such that

$$\mathbf{u} = \beta\tilde{\mathbf{u}}(\alpha, \beta\mathbf{x} + \gamma).$$

Equivalence group of mKdV-like equations

[R.O. Popovych, O.O. Vaneeva, Commun Nonlinear Sci Numer Simulat 15 (2010) 3887–3899]

The variable-coefficient mKdV-like equations

$$u_t + f(t)u^2u_x + g(t)u_{xxx} + h(t)u + (p(t) + q(t)x)u_x + k(t)uu_x + l(t) = 0$$

admit the equivalence group consisting of the transformations

$$\tilde{t} = \alpha(t), \quad \tilde{x} = \beta(t)x + \gamma(t), \quad \tilde{u} = \theta(t)u + \psi(t),$$

where α, β, γ, θ and ψ run through the set of smooth functions of t, $\alpha\beta\theta \neq 0$.

Equivalence group of mKdV-like equations

The arbitrary elements of the equations

$$u_t + f(t)u^2u_x + g(t)u_{xxx} + h(t)u + (p(t) + q(t)x)u_x + k(t)uu_x + l(t) = 0$$

are transformed by the formulas

$$\tilde{f} = \frac{\beta}{\alpha_t\theta^2}f, \quad \tilde{g} = \frac{\beta^3}{\alpha_t}g, \quad \tilde{h} = \frac{1}{\alpha_t}\left(h - \frac{\theta_t}{\theta}\right), \quad \tilde{q} = \frac{1}{\alpha_t}\left(q + \frac{\beta_t}{\beta}\right),$$

$$\tilde{p} = \frac{1}{\alpha_t}\left(\beta p - \gamma q + \beta\frac{\psi^2}{\theta^2}f - \beta\frac{\psi}{\theta}k + \gamma_t - \gamma\frac{\beta_t}{\beta}\right),$$

$$\tilde{k} = \frac{\beta}{\alpha_t\theta}\left(k - 2\frac{\psi}{\theta}f\right), \quad \tilde{l} = \frac{1}{\alpha_t}\left(\theta l - \psi h - \psi_t + \psi\frac{\theta_t}{\theta}\right).$$

Five of the arbitrary elements can be gauged to simple constant values! For example, it is possible to set $g = 1$ and $h = p = q = l = 0$.

Criterion of reducibilty

An equation of the form

$$u_t + f(t)u^2u_x + g(t)u_{xxx} + h(t)u + (p(t) + q(t)x)u_x + k(t)uu_x + l(t) = 0$$

is similar to the standard mKdV equation

$$u_t + u^2u_x + u_{xxx} = 0$$

if and only if

$$2h - 2q = \frac{f_t}{f} - \frac{g_t}{g}, \quad 2lf = k_t + kh - k\frac{f_t}{f}.$$

$$u_t + f\,u^2u_x + g\,u_{xxx} + h\,u + (p + q\,x)u_x + k\,uu_x + \frac{1}{2f}\left(k_t + kh - k\frac{f_t}{f}\right) = 0$$

$g(t)$	Basis of $A^{\max}$
$\forall$	$e^{\int qdt}\partial_x$
$c_0 f e^{2\int(q-h)dt}\left(\frac{H}{F}\right)^n$	$e^{\int qdt}\partial_x,\ H\partial_t + \left[(qH + 2n + 2)x + H\left(p - \frac{k^2}{4f}\right) - 2(n+1)Q\right]\partial_x + \left[(n - 2 - hH)u + \frac{k}{2f}(n-2) - lH\right]\partial_u$
$c_0 f e^{\int\left(mfe^{-\int(q+2h)\,dt}\right)dt}$	$e^{\int qdt}\partial_x,\ F\partial_t + \left[(qF + 2m)x + F\left(p - \frac{k^2}{4f}\right) - 2mQ\right]\partial_x + \left[(m - hF)u + \frac{m}{2}\frac{k}{f} - lF\right]\partial_u$
$c_0 f e^{2\int(q-h)dt}$	$e^{\int qdt}\partial_x,\ F\left[\partial_t + \left(qx + p - \frac{k^2}{4f}\right)\partial_x - (hu + l)\,\partial_u\right],\ H\partial_t + \left[(qH + 2)x + H\left(p - \frac{k^2}{4f}\right) - 2Q\right]\partial_x - \left[(2 + hH)u + \frac{k}{f} + lH\right]\partial$

The functions f, h, p, q and k are arbitrary functions of the variable t in all cases, $f \neq$ c_0, c_1, m and n are arbitrary constants, $c_0mn \neq 0$,

$$F = \frac{6}{f}\,e^{\int(q+2h)dt}, \quad H = F\left(\int fe^{-\int(q+2h)\,dt}dt + c_1\right),$$
$$Q = e^{\int qdt}\int\left(p - \frac{k^2}{4f}\right)e^{-\int qdt}dt, \quad l = \frac{1}{2f}\left(k_t + kh - k\frac{f_t}{f}\right).$$

Conclusion

Point transformations can essentially simplify a wide range of problems of mathematical physics, where the object of the study is a class of differential equation.
These are, in particular,

- group classification problem
- studying of conditional, potential and other kinds of symmetries
- finding of exact solutions
- construction of conservation laws
- investigation of integrability
- ...

Thank you for your attention!