ON GEOMETRIC SPINORS AND THEIR TRANSFORMATIONS PROPERTIES

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Introduction

We will follow the approach in which spinors are constructed in terms of nilpotents formed from the spacetime basis vectors represented as generators of the Clifford algebra $Cl(1,3)$.

$$\gamma_a \cdot \gamma_b \equiv \tfrac{1}{2}(\gamma_a \gamma_b + \gamma_b \gamma_a) = \eta_{ab}$$

$$\gamma_a \wedge \gamma_b \equiv \tfrac{1}{2}(\gamma_a \gamma_b - \gamma_b \gamma_a)$$

The inner, symmetric, product of basis vectors γ_a gives the metric, η_{ab} .

The outer, antisymmetric, product of basis vectors gives the basis bivector.

Generic Clifford number

$$\Phi = \phi^A \gamma_A$$

where

$$\gamma_A \equiv \gamma_{a_1 a_2 \ldots a_r} \equiv \gamma_{a_1} \wedge \gamma_{a_2} \wedge \ldots \wedge \gamma_{a_r}$$

$$r = 0, 1, 2, 3, 4$$

Spinors are particular Clifford numbers

$$\Psi = \psi^\alpha \xi_\alpha$$

where ξ_α are spinor basis elements, composed from γ_A .

We will consider transformation properties of Clifford numbers.

In general, a Clifford number transforms according to

(1) $$\Phi \rightarrow \Phi' = \mathrm{R}\,\Phi\,\mathrm{S}$$

Clifford numbers

e.g., $\mathrm{R} = \mathrm{e}^{\frac{1}{2}\alpha^A \gamma_A}$, $\mathrm{S} = \mathrm{e}^{\frac{1}{2}\beta^A \gamma_A}$

In particular, if $\mathrm{S} = 1$, we have

$$\Phi \rightarrow \Phi' = \mathrm{R}\,\Phi$$

As an example, let us consider the case

$$\mathrm{R} = \mathrm{e}^{\frac{1}{2}\alpha\gamma_1\gamma_2} = \cos\frac{\alpha}{2} + \gamma_1\gamma_2 \sin\frac{\alpha}{2}, \qquad \mathrm{S} = \mathrm{e}^{\frac{1}{2}\beta\gamma_1\gamma_2} = \cos\frac{\beta}{2} + \gamma_1\gamma_2 \sin\frac{\beta}{2}$$

and examine, how various Clifford numbers, $X = X^C \gamma_C$, transform under (1), which now reads: $X \rightarrow X' = \mathrm{R}\,X\,\mathrm{S}$

(i) If $X = X^1 \gamma_1 + X^2 \gamma_2$

then

$$X' = X^1 \left(\gamma_1 \cos\frac{\alpha - \beta}{2} + \gamma_2 \sin\frac{\alpha - \beta}{2} \right) + X^2 \left(-\gamma_1 \sin\frac{\alpha - \beta}{2} + \gamma_2 \cos\frac{\alpha - \beta}{2} \right)$$

(ii) $$X = X^3\gamma_3 + X^{123}\gamma_{123}$$

$$X' = X^3\left(\gamma_3 \cos\frac{\alpha+\beta}{2} + \gamma_{123}\sin\frac{\alpha+\beta}{2}\right) + X^{123}\left(-\gamma_2 \sin\frac{\alpha+\beta}{2} + \gamma_{123}\cos\frac{\alpha+\beta}{2}\right)$$

(iii) $$X = s\,\underline{1} + X^{12}\gamma_{12}$$

$$X' = s\left(\underline{1}\cos\frac{\alpha+\beta}{2} + \gamma_{12}\sin\frac{\alpha+\beta}{2}\right) + X^2\left(-\underline{1}\sin\frac{\alpha+\beta}{2} + \gamma_{12}\cos\frac{\alpha+\beta}{2}\right)$$

(iv) $$X = \tilde{X}^1\gamma_5\gamma_1 + \tilde{X}^2\gamma_5\gamma_2$$

$$X' = \tilde{X}^1\left(\gamma_5\gamma_1\cos\frac{\alpha-\beta}{2} + \gamma_5\gamma_2\sin\frac{\alpha-\beta}{2}\right) + \tilde{X}^2\left(-\gamma_5\gamma_1\sin\frac{\alpha-\beta}{2} + \gamma_5\gamma_2\cos\frac{\alpha-\beta}{2}\right)$$

Usual rotations of vectors or pseudovectors are reproduced, if the angle β for the right transformation is equal to minus angle α for the left transformation, i.e., if

$$\beta = -\alpha$$

Then all other transformations which mix the grade vanish.

Clifford algebra and spinors in Minkowski space

$$\gamma_a = (\gamma_0, \gamma_1, \gamma_2, \gamma_3)$$

Witt basis

$$\theta_1 = \frac{1}{2}(\gamma_0 + \gamma_3), \quad \theta_2 = \frac{1}{2}(\gamma_1 + i\gamma_2),$$

$$\bar{\theta}_1 = \frac{1}{2}(\gamma_0 - \gamma_3), \quad \bar{\theta}_2 = \frac{1}{2}(\gamma_1 - i\gamma_2)$$

The new basis vectors satisfy

$$\{\theta_a, \bar{\theta}_b\} = \eta_{ab}, \quad \{\theta_a, \theta_b\} = 0, \quad \{\bar{\theta}_a, \bar{\theta}_b\} = 0$$

Fermionic anticommutation relations

We now observe that the product

$$f = \bar{\theta}_1 \bar{\theta}_2$$

satisfies

$$\bar{\theta}_a f = 0, \qquad a = 1,2$$

f can be interpreted as `vacuum', and $\bar{\theta}_a$ can be interpreted as operators that annihilate f.

An object constructed as a superposition

$$\Psi = (\psi^0 \underline{1} + \psi^1 \theta_1 + \psi^2 \theta_2 + \psi^{12} \theta_1 \theta_2) f$$

is a 4-component spinor.

It is convenient to change the notation:

$$\Psi = (\psi^1 \underline{1} + \psi^2 \theta_1\theta_2 + \psi^3 \theta_1 + \psi^4 \theta_2) f = \psi^\alpha \xi_\alpha \,, \qquad \alpha = 1,2,3,4$$

Spinor basis

Even part $\Psi_L = (\psi^1 \underline{1} + \psi^2 \theta_1\theta_2)\,\bar\theta_1\bar\theta_2$

Odd part $\Psi_R = (\psi^3 \theta_1 + \psi^4 \theta_2)\bar\theta_1\bar\theta_2$

$$i\gamma_5 \Psi_L = -\Psi_L$$

$$i\gamma_5 \Psi_R = \Psi_R$$

$$\gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3$$

Under the transformations

$$\Psi \to \Psi' = \mathrm{R}\Psi \,, \qquad \mathrm{R} = \exp[\tfrac{1}{2}\gamma_{a_1}\gamma_{a_2}\varphi]$$

Ψ transforms as a Dirac spinor.

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$$f = \bar{\theta}_1 \bar{\theta}_2$$

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Example:

$$\mathrm{R} = \mathrm{e}^{\frac{1}{2}\gamma_1\gamma_2\varphi} = \cos\frac{\varphi}{2} + \gamma_1\gamma_2 \sin\frac{\varphi}{2}$$

$$\Psi \to \Psi' = \mathrm{R}\Psi = \left(\mathrm{e}^{\frac{i\varphi}{2}} \psi^1 \underline{1} + \mathrm{e}^{-\frac{i\varphi}{2}} \psi^2 \theta_1\theta_2 + \mathrm{e}^{\frac{i\varphi}{2}} \psi^3 \theta_1 + \mathrm{e}^{-\frac{i\varphi}{2}} \psi^4 \theta_2 \right) f$$

This is the well-known transformation of a 4-component spinor.

Four independent spinors

Four different possible vacua:

$$f_1 = \bar{\theta}_1 \bar{\theta}_2\,, \qquad f_2 = \theta_1 \theta_2\,, \qquad f_3 = \theta_1 \bar{\theta}_2\,, \qquad f_3 = \bar{\theta}_1 \theta_2$$

Four different kinds of spinors:

$$\Psi^1 = (\psi^{11}\,\underline{1} + \psi^{21}\theta_1\theta_2 + \psi^{31}\theta_1 + \psi^{41}\theta_2)\,f_1$$

$$\Psi^2 = (\psi^{12}\,\underline{1} + \psi^{22}\bar{\theta}_1\bar{\theta}_2 + \psi^{32}\bar{\theta}_1 + \psi^{42}\bar{\theta}_2)\,f_2$$

$$\Psi^3 = (\psi^{13}\bar{\theta}_1 + \psi^{23}\theta_2 + \psi^{33}\,\underline{1} + \psi^{43}\bar{\theta}_1\theta_2)\,f_3$$

$$\Psi^4 = (\psi^{14}\theta_1 + \psi^{24}\bar{\theta}_2 + \psi^{34}\,\underline{1} + \psi^{44}\theta_1\bar{\theta}_2)\,f_4$$

Each of those spinors lives in a different minimal left ideal of $Cl(1,3)$.

In general, complexified version

An arbitrary element of $Cl(1,3)$ is the sum:

$$\Phi = \Psi^1 + \Psi^2 + \Psi^3 + \Psi^4 = \psi^{\alpha i}\xi_{\alpha i} \equiv \psi^{\tilde{A}}\xi_{\tilde{A}}$$

`Generalized spinor'

$$\alpha = 1,2,3,4; \quad i = 1,2,3,4$$

$$\xi_{\tilde{A}} \equiv \xi_{\alpha i} = \{\underline{1}\,f_1,\ \theta_1\theta_2 f_1, \ldots, \theta_1 f_4,\ \bar{\theta}_2 f_4,\ \underline{1}\,f_4,\ \bar{\theta}_1\theta_2 f_4\},$$

Spinor basis of $Cl(1,3)$

Four independent spinors

Four different possible vacua:

$$f_1 = \bar{\theta}_1\bar{\theta}_2\,, \qquad f_2 = \theta_1\theta_2\,, \qquad f_3 = \theta_1\bar{\theta}_2\,, \qquad f_3 = \bar{\theta}_1\theta_2$$

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$\alpha = 1,2,3,4; \quad i = 1,2,3,4$

$$\xi_{\tilde{A}} \equiv \xi_{\alpha i} = \{\underline{1}f_1,\, \theta_1\theta_2 f_1, \ldots, \theta_1 f_4,\, \bar{\theta}_2 f_4,\, \underline{1}f_4,\, \bar{\theta}_1\theta_2 f_4\},$$

Spinor basis of $Cl(1,3)$

Matrix notation:

$$\psi^{\alpha i} = \begin{pmatrix} \psi^{11} & \psi^{12} & \psi^{13} & \psi^{14} \\ \psi^{21} & \psi^{22} & \psi^{23} & \psi^{24} \\ \psi^{31} & \psi^{32} & \psi^{33} & \psi^{34} \\ \psi^{41} & \psi^{42} & \psi^{43} & \psi^{44} \end{pmatrix}, \qquad \xi_{\tilde{A}} \equiv \xi_{\alpha i} = \begin{pmatrix} f_1 & f_2 & \bar{\theta}_1 f_3 & \theta_1 f_4 \\ \theta_1\theta_2 f_1 & \bar{\theta}_1\bar{\theta}_2 f_2 & \theta_2 f_3 & \bar{\theta}_2 f_4 \\ \theta_1 f_1 & \bar{\theta}_1 f_2 & f_3 & f_4 \\ \theta_2 f_1 & \bar{\theta}_2 f_2 & \bar{\theta}_1\theta_2 f_3 & \theta_1\bar{\theta}_2 f_4 \end{pmatrix}$$

Four independent spinors

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$$f_1 = \bar{\theta}_1 \bar{\theta}_2 , \qquad f_2 = \theta_1 \theta_2 , \qquad f_3 = \theta_1 \bar{\theta}_2 , \qquad f_3 = \bar{\theta}_1 \theta_2$$

Four different kinds of spinors:

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In general, complexified version

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$\alpha = 1,2,3,4; \quad i = 1,2,3,4$

$$\xi_{\tilde{A}} \equiv \xi_{\alpha i} = \{ \underline{1} f_1, \, \theta_1 \theta_2 f_1, \ldots, \theta_1 f_4, \, \bar{\theta}_2 f_4, \, \underline{1} f_4, \, \bar{\theta}_1 \theta_2 f_4 \},$$

Spinor basis of $Cl(1,3)$

Matrix notation:

$$\psi^{\alpha i} = \begin{pmatrix} \psi^{11} & \psi^{12} & \psi^{13} & \psi^{14} \\ \psi^{21} & \psi^{22} & \psi^{23} & \psi^{24} \\ \psi^{31} & \psi^{32} & \psi^{33} & \psi^{34} \\ \psi^{41} & \psi^{42} & \psi^{43} & \psi^{44} \end{pmatrix}, \qquad \xi_{\tilde{A}} \equiv \xi_{\alpha i} = \begin{pmatrix} f_1 & f_2 & \bar{\theta}_1 f_3 & \theta_1 f_4 \\ \theta_1 \theta_2 f_1 & \bar{\theta}_1 \bar{\theta}_2 f_2 & \theta_2 f_3 & \bar{\theta}_2 f_4 \\ \theta_1 f_1 & \bar{\theta}_1 f_2 & f_3 & f_4 \\ \theta_2 f_1 & \bar{\theta}_2 f_2 & \bar{\theta}_1 \theta_2 f_3 & \theta_1 \bar{\theta}_2 f_4 \end{pmatrix}$$

Spinor of the 2nd left ideal

Basis of the 2nd left ideal

A general transformation is

$$\Phi = \psi^{\tilde{A}} \xi_{\tilde{A}} \rightarrow \Phi' = \mathrm{R}\,\Phi\,\mathrm{S} = \psi^{\tilde{A}} \xi'_{\tilde{A}} = \psi^{A} L_{A}{}^{B} \xi_{B} = \psi'^{B} \xi_{B}$$

Active transformation

where $\xi'_{\tilde{A}} = \mathrm{R}\,\xi_{\tilde{A}}\,\mathrm{S} = L_{\tilde{A}}{}^{\tilde{B}} \xi_{\tilde{B}}$, $\qquad \psi'^{\tilde{B}} = \psi^{\tilde{A}} L_{\tilde{A}}{}^{\tilde{B}}$

$$\tilde{A} \equiv \alpha i$$

$$\alpha = 1, 2, 3, 4$$

$$i = 1, 2, 3, 4$$

The transformation from the left,

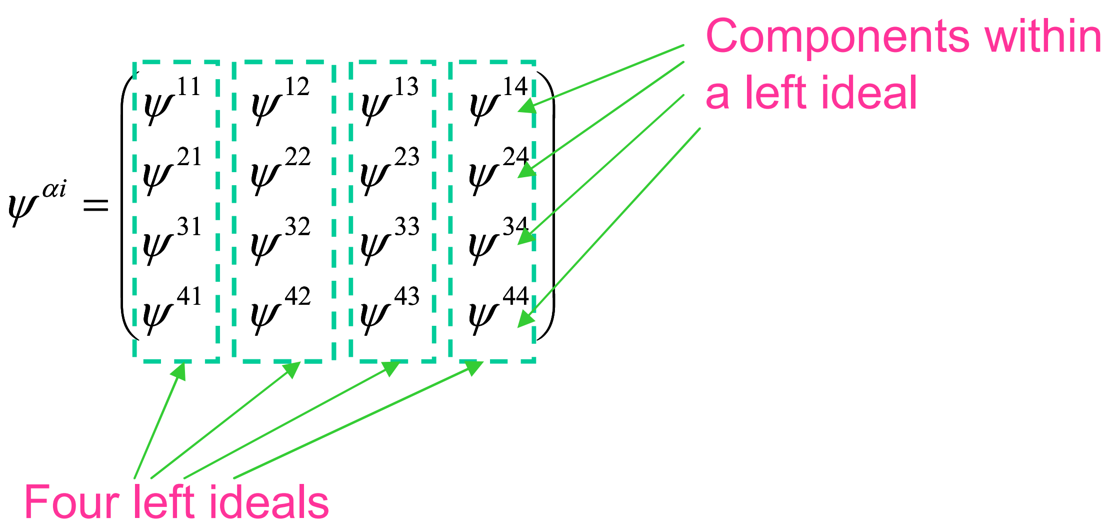
$$\Phi' = \mathrm{R}\,\Phi$$

reshuffles the components within each left ideal,
whereas the transformation from the right,

$$\Phi' = \Phi\,\mathrm{S}$$

reshuffles the left ideals.

$$\psi^{\alpha i} = \begin{pmatrix} \psi^{11} & \psi^{12} & \psi^{13} & \psi^{14} \\ \psi^{21} & \psi^{22} & \psi^{23} & \psi^{24} \\ \psi^{31} & \psi^{32} & \psi^{33} & \psi^{34} \\ \psi^{41} & \psi^{42} & \psi^{43} & \psi^{44} \end{pmatrix}$$



Passive transformations

(2) $$\Phi' = \psi'^{\tilde{A}} \xi'_{\tilde{A}} = \psi^{\tilde{A}} \xi_{\tilde{A}} = \Phi$$

The object remains the same

If the spinor basis transforms according to

(3) $$\xi'_{\tilde{A}} = \mathrm{R} \xi_{\tilde{A}} \mathrm{S} = L_{\tilde{A}}^{\ \tilde{B}} \xi_{\tilde{B}}$$

then the components must transform as

(4) $$\psi'^{\tilde{A}} = \psi^{\tilde{B}} (L^{-1})_{\tilde{B}}^{\ \tilde{A}}$$

With respect to the new basis, $\xi'_{\tilde{A}}$, (new reference frame), the generalized spinor, Φ, has transformed components.

From (2) – (4) we obtain

$$\psi^{\tilde{B}} (L^{-1})_{\tilde{B}}^{\ \tilde{A}} \xi'_{\tilde{A}} = \psi^{\tilde{B}} \mathrm{R}^{-1} \xi'_{\tilde{B}} \mathrm{S}^{-1} = \psi^{\tilde{B}} \xi_{\tilde{B}}$$

This is the active transformation of the object $\psi^{\tilde{B}} \xi'_{\tilde{B}}$:

$$\psi^{\tilde{B}} \xi'_{\tilde{B}} \rightarrow \psi^{\tilde{B}} \xi_{\tilde{B}} = \psi^{\tilde{B}} \mathrm{R}^{-1} \xi'_{\tilde{B}} \mathrm{S}^{-1}$$

This is equivalent to the active transformation of the object $\psi^{\tilde{B}} \xi_{\tilde{B}}$:

$$\psi^{\tilde{B}} \xi_{\tilde{B}} \rightarrow \psi^{\tilde{B}} \xi'_{\tilde{B}} = \psi^{\tilde{B}} \mathrm{R} \xi_{\tilde{B}} \mathrm{S}$$

Active transformations are thus embedded in passive transformations.

Behavior of spinors under Lorentz transformations

Rotated object

$a = 0, 1, 2, 3$

Let us consider the following transformation of the basis vectors

(5) $$\gamma_a \to \gamma'_a = \mathrm{R}\, \gamma_a \mathrm{R}^{-1}$$

R is a proper or improper Lorentz transformation

A generalized spinor, $\Phi \in Cl(1,3)$, composed of γ_a, then transforms according to

$$\Phi = \psi^{\tilde{A}} \xi_{\tilde{A}} \to \Phi' = \psi^{\tilde{A}} \xi'_{\tilde{A}} = \psi^{A} \mathrm{R}\, \xi_B \mathrm{R}^{-1} = \mathrm{R}\, \Phi\, \mathrm{R}^{-1}$$

The transformation (5) of the basis vectors has for a consequence that the object Φ does not transform only from the right, but also from the left.

Piazzese 1993: Spinors cannot be interpreted as the minimal ideals of Clifford algebras

Behavior of spinors under Lorentz transformations

Rotated object

Let us consider the following transformation of the basis vectors

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A generalized spinor, $\Phi \in Cl(1,3)$, composed of γ_a, then transforms according to

$$\Phi = \psi^{\tilde{A}} \xi_{\tilde{A}} \to \Phi' = \psi^{\tilde{A}} \xi'_{\tilde{A}} = \psi^{A} R\, \xi_B R^{-1} = R\,\Phi\, R^{-1}$$

The transformation (2) of the basis vectors has for a consequence that the object Φ does not transform only from the right, but also from the left.

Piazzese 1993: Spinors cannot be interpreted as the minimal ideals of Clifford algebras

But: If the reference frame transforms as

$$\gamma_a \to \gamma'_a = R\,\gamma_a$$

then

$$\Phi = \psi^{\tilde{A}} \xi_{\tilde{A}} \to \Phi' = \psi^{\tilde{A}} \xi'_{\tilde{A}} = \psi^{A} R\, \xi_B = R\,\Phi$$

Transformation of a spinor

The ideal approach is OK

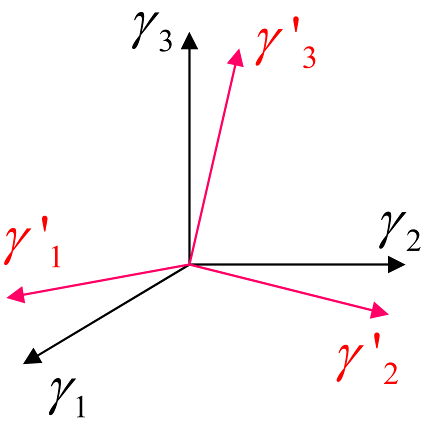
Usually, reference frames are ``rotated'' (Lorentz rotated) according to

$$\gamma_a \rightarrow \gamma'_a = \mathrm{R}\,\gamma_a \mathrm{R}^{-1} = L_a{}^b \gamma_b$$

A proper or improper Lorentz transformation

Therefore, a ``rotated'' observer sees (generalized) spinors transformed according to

$$\Phi \rightarrow \Phi' = \mathrm{R}\,\Phi\,\mathrm{R}^{-1}$$



With respect to a new reference frame, the object

$$\Phi = \psi^{\tilde{A}} \xi_{\tilde{A}}$$

is expanded as

$$\Phi = \psi'^{\tilde{A}} \xi'_{\tilde{A}}$$

where $\psi'^{\tilde{A}} = \psi^{\tilde{B}} (L^{-1})_{\tilde{B}}{}^{\tilde{A}}$

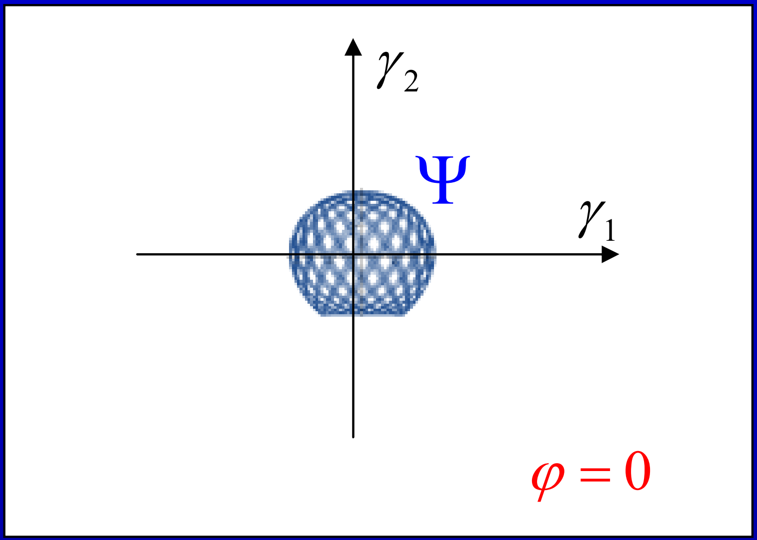
The corresponding matrix $\psi^{\alpha i}$ transforms from the left and from the right.

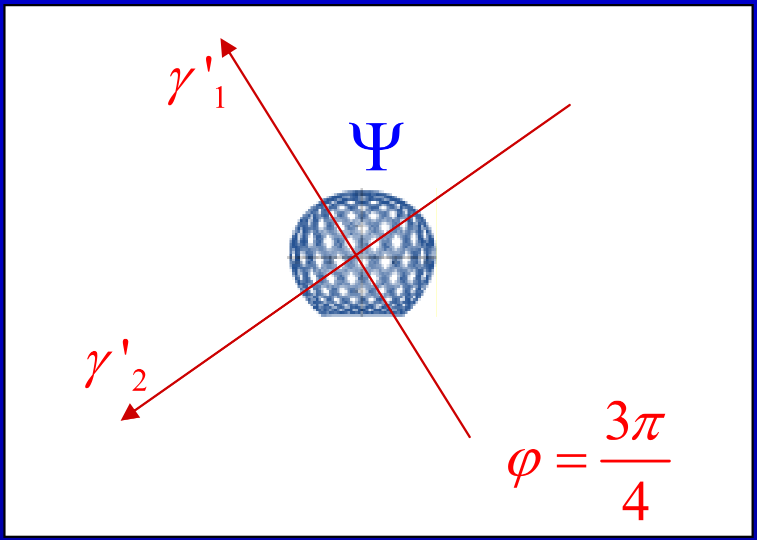
$$\tilde{A} \equiv \alpha i\,, \quad B \equiv \beta j$$

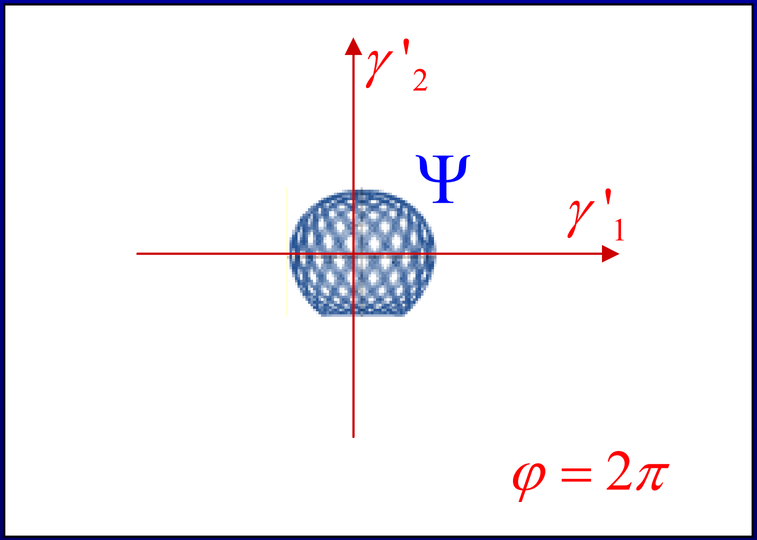
$$\alpha, \beta = 1,2,3,4$$

$$i, j = 1,2,3,4$$

Rotation in the (γ_1, γ_2) plane



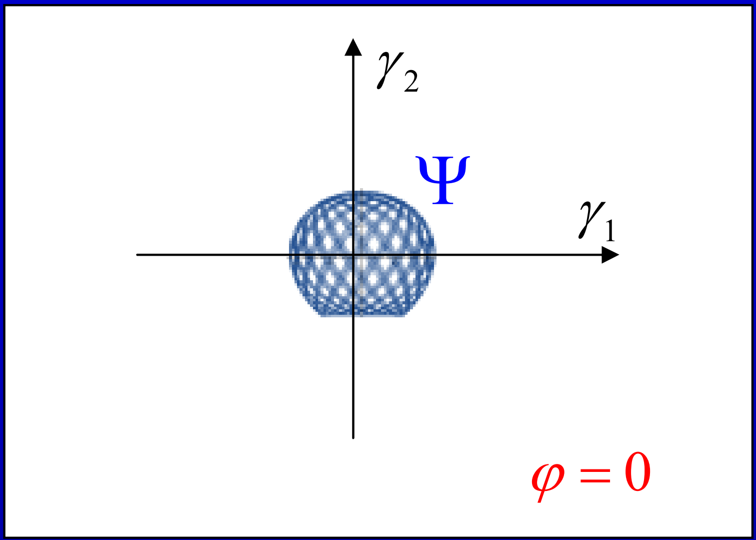


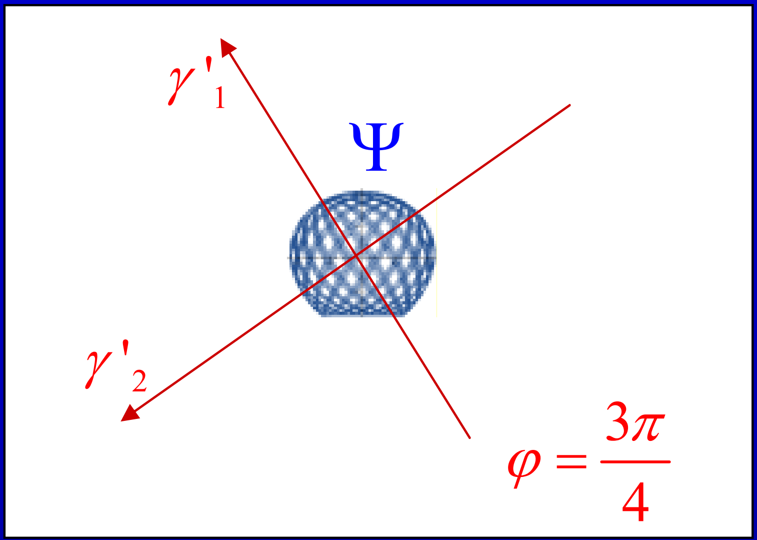


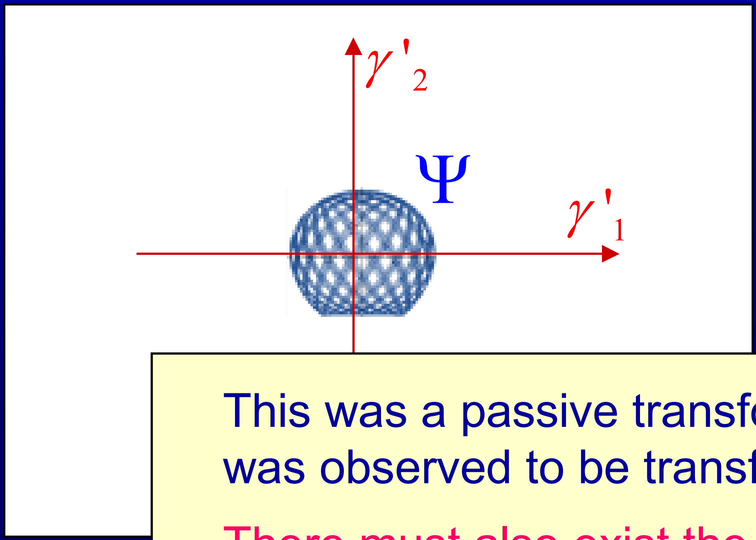
If the observer, together with the reference frame, starts to rotate, then after having exhibited the $\varphi = 2\pi$ turn, he observes the same spinor Ψ , as he did at $\varphi = 0$.

The sign of the spinor did not change.

Rotation in the (γ_1, γ_2) plane







If the observer, together with the reference frame, starts to rotate, then after having exhibited the $\varphi = 2\pi$ turn, he observes the same spinor Ψ, as he did at $\varphi = 0$.

The sign of the spinor did not change.

This was a passive transformation. In the new reference frame the object was observed to be transformed according to $\Psi' = R\,\Psi R^{-1}$.

There must also exist the corresponding active transformation, such that in a fixed reference frame the spinor Ψ transform as $\Psi' = R\,\Psi R^{-1}$.

Examples

1) Rotation

$$\gamma_0 \to \gamma_0\,, \quad \gamma_1 \to \gamma_1\,, \quad \gamma_2 \to \gamma_2 \cos\vartheta + \gamma_3 \sin\vartheta$$
$$\gamma_3 \to -\gamma_2 \sin\vartheta + \gamma_3 \cos\vartheta$$

Case $\vartheta = \pi$: $\gamma_0 \to \gamma_0\,, \quad \gamma_1 \to \gamma_1\,, \quad \gamma_2 \to -\gamma_2\,, \quad \gamma_3 \to -\gamma_3$

$$\theta_1 \to \bar\theta_1,$$
$$\theta_2 \to \bar\theta_2$$
$$\bar\theta_1 \to \theta_1$$
$$\bar\theta_2 \to \theta_2$$

This is a spinor of the 2nd left ideal

A spinor of the first left ideal transforms as

$$(\underbrace{\psi^{11}\underline{1} + \psi^{21}\theta_1\theta_2}_{L} + \underbrace{\psi^{31}\theta_1 + \psi^{41}\theta_2}_{R})\,\bar\theta_1\bar\theta_2 \to (\underbrace{\psi^{11}\underline{1} + \psi^{21}\bar\theta_1\bar\theta_2}_{L} + \underbrace{\psi^{31}\bar\theta_1 + \psi^{41}\bar\theta_2}_{R})\,\theta_1\theta_2$$

A left handed spinor of the *first ideal* transforms into a left handed spinor of the *second ideal*.

Under the $\vartheta = \pi$ rotation in the (γ_2, γ_3) plane, a generalized spinor

$$\begin{aligned}\Phi = &(\psi^{11}\underline{1} + \psi^{21}\theta_1\theta_2 + \psi^{31}\theta_1 + \psi^{41}\theta_2)\bar{\theta}_1\bar{\theta}_2 \\ &+ (\psi^{12}\underline{1} + \psi^{22}\bar{\theta}_1\bar{\theta}_2 + \psi^{32}\bar{\theta}_1 + \psi^{42}\bar{\theta}_2)\theta_1\theta_2 \\ &+ (\psi^{13}\bar{\theta}_1 + \psi^{23}\theta_2 + \psi^{33}\underline{1} + \psi^{43}\bar{\theta}_1\theta_2)\theta_1\bar{\theta}_2 \\ &+ (\psi^{14}\theta_1 + \psi^{24}\bar{\theta}_2 + \psi^{34}\underline{1} + \psi^{44}\theta_1\bar{\theta}_2)\bar{\theta}_1\theta_2\end{aligned}$$

transforms into

$$\begin{aligned}\Phi' = &(\psi^{11}\underline{1} + \psi^{21}\bar{\theta}_1\bar{\theta}_2 + \psi^{31}\bar{\theta}_1 + \psi^{41}\bar{\theta}_2)\theta_1\theta_2 \\ &+ (\psi^{12}\underline{1} + \psi^{22}\theta_1\theta_2 + \psi^{32}\theta_1 + \psi^{42}\theta_2)\bar{\theta}_1\bar{\theta}_2 \\ &+ (\psi^{13}\theta_1 + \psi^{23}\bar{\theta}_2 + \psi^{33}\underline{1} + \psi^{43}\theta_1\bar{\theta}_2)\bar{\theta}_1\theta_2 \\ &+ (\psi^{14}\bar{\theta}_1 + \psi^{24}\theta_2 + \psi^{34}\underline{1} + \psi^{44}\bar{\theta}_1\theta_2)\theta_1\bar{\theta}_2\end{aligned}$$

The matrix of components

$$\psi^{\alpha i} = \begin{pmatrix} \psi^{11} & \psi^{12} & \psi^{13} & \psi^{14} \\ \psi^{21} & \psi^{22} & \psi^{23} & \psi^{24} \\ \psi^{31} & \psi^{32} & \psi^{33} & \psi^{34} \\ \psi^{41} & \psi^{42} & \psi^{43} & \psi^{44} \end{pmatrix}$$

transforms into

$$\psi'^{\alpha i} = \begin{pmatrix} \psi^{12} & \psi^{11} & \psi^{14} & \psi^{13} \\ \psi^{22} & \psi^{21} & \psi^{24} & \psi^{23} \\ \psi^{32} & \psi^{31} & \psi^{34} & \psi^{33} \\ \psi^{42} & \psi^{41} & \psi^{44} & \psi^{43} \end{pmatrix}$$

The spinor of the 1st ideal transforms into the spinor of the 2nd ideal

$$\xi_{11} = \bar{\theta}_1\bar{\theta}_2\,, \qquad \xi_{21} = \theta_1\theta_2\bar{\theta}_1\bar{\theta}_2$$

spinor basis states of the 1st left ideal

$$\xi_{12} = \theta_1\theta_2\,, \qquad \xi_{22} = \bar{\theta}_1\bar{\theta}_2\theta_1\theta_2$$

spinor basis states of the 2nd left ideal

Rotation $\gamma_0 \to \gamma_0\,, \quad \gamma_1 \to \gamma_1\,, \quad \gamma_2 \to -\gamma_2\,, \quad \gamma_3 \to -\gamma_3$

gives $\theta_1 \to \bar{\theta}_1\,, \quad \theta_2 \to \bar{\theta}_2\,, \quad \bar{\theta}_1 \to \theta_1\,, \quad \bar{\theta}_2 \to \theta_2$

Therefore, the spinor basis states transform as

$$\xi_{11} \to \xi_{12}\,, \qquad \xi_{21} \to \xi_{22}$$
$$\xi_{12} \to \xi_{11}\,, \qquad \xi_{22} \to \xi_{21}$$

Under the $\vartheta = \pi$ rotation, the spin $1/2$ state of the 1st ideal transforms into the spin $-1/2$ state of the 2nd ideal, and vice versa.

They are eigenvalues of the spin operator $-\frac{i}{2}\gamma_1\gamma_2$

$$-\frac{i}{2}\gamma_1\gamma_2\,\xi_{11} = \frac{1}{2}\xi_{11}\,, \qquad -\frac{i}{2}\gamma_1\gamma_2\,\xi_{21} = -\frac{1}{2}\xi_{21}$$

$$-\frac{i}{2}\gamma_1\gamma_2\,\xi_{12} = -\frac{1}{2}\xi_{12}\,, \qquad -\frac{i}{2}\gamma_1\gamma_2\,\xi_{22} = \frac{1}{2}\xi_{22}$$

New basis states

$$\xi^1_{1/2} = \frac{1}{\sqrt{2}}(\xi_{11} + \xi_{22}), \qquad \xi^2_{1/2} = \frac{1}{\sqrt{2}}(\xi_{11} - \xi_{22})$$

$$\xi^1_{-1/2} = \frac{1}{\sqrt{2}}(\xi_{21} + \xi_{12}), \quad \xi^2_{-1/2} = \frac{1}{\sqrt{2}}(\xi_{21} - \xi_{12})$$

A superposition of the states of the 1st and the 2nd ideal

Under the rotation $\gamma_0 \to \gamma_0, \quad \gamma_1 \to \gamma_1, \quad \gamma_2 \to -\gamma_2, \quad \gamma_3 \to -\gamma_3$

we have

$$\xi^1_{1/2} \to \frac{1}{\sqrt{2}}(\xi_{12} + \xi_{21}) = \xi^1_{-1/2}$$

$$\xi^1_{-1/2} \to \frac{1}{\sqrt{2}}(\xi_{22} + \xi_{11}) = \xi^1_{1/2}$$

$$\xi^2_{1/2} \to \frac{1}{\sqrt{2}}(\xi_{12} - \xi_{21}) = -\xi^2_{-1/2}$$

$$\xi^2_{-1/2} \to \frac{1}{\sqrt{2}}(\xi_{22} - \xi_{11}) = -\xi^2_{1/2}$$

$$-\frac{i}{2}\gamma_1\gamma_2\xi^1_{\pm 1/2} = \pm\frac{1}{2}\xi^1_{\pm 1/2}$$

$$-\frac{i}{2}\gamma_1\gamma_2\xi^2_{\pm 1/2} = \pm\frac{1}{2}\xi^2_{\pm 1/2}$$

These states have definite spin projection.
Under the $\vartheta = \pi$ rotation, the spin $1/2$ state transforms into the spin $-1/2$ state, and vice versa.

2) Space inversion

$$\gamma_0 \to \gamma'_0 = \gamma_0, \quad \gamma_r \to \gamma'_r = -\gamma_r, \ r = 1,2,3$$

$$\theta_1 \to \tfrac{1}{2}(\gamma_0 - \gamma_3) = \bar{\theta}_1,$$

$$\theta_2 \to \tfrac{1}{2}(-\gamma_1 - i\gamma_2) = -\theta_2$$

$$\bar{\theta}_1 \to \tfrac{1}{2}(\gamma_0 + \gamma_3) = \theta_1$$

$$\bar{\theta}_2 \to \tfrac{1}{2}(-\gamma_1 + i\gamma_2) = -\bar{\theta}_2$$

A spinor of the first left ideal transforms as

This is a spinor of the 3rd left ideal

$$(\underbrace{\psi^{11}\underline{1} + \psi^{21}\theta_1\theta_2}_{L} + \underbrace{\psi^{31}\theta_1 + \psi^{41}\theta_2}_{R})\bar{\theta}_1\bar{\theta}_2 \to (-\underbrace{\psi^{11}1 + \psi^{21}\bar{\theta}_1\theta_2}_{R} - \underbrace{\psi^{31}\bar{\theta}_1 + \psi^{41}\theta_2}_{L})\theta_1\bar{\theta}_2$$

A left handed spinor of the *first ideal* transforms into a right handed spinor of the *third ideal*.

In general, under *space inversion*, the matrix of the spinor basis elements

$$\xi_{\alpha i} = \begin{pmatrix} f_1 & f_2 & \bar{\theta}_1 f_3 & \theta_1 f_4 \\ \theta_1\theta_2 f_1 & \bar{\theta}_1\bar{\theta}_2 f_2 & \theta_2 f_3 & \bar{\theta}_2 f_4 \\ \theta_1 f_1 & \bar{\theta}_1 f_2 & f_3 & f_4 \\ \theta_2 f_1 & \bar{\theta}_2 f_2 & \bar{\theta}_1\theta_2 f_3 & \theta_1\bar{\theta}_2 f_4 \end{pmatrix}$$

transforms into

$$\xi'_{\alpha i} = \begin{pmatrix} -f_3 & -f_4 & -\theta_1 f_1 & -\bar{\theta}_1 f_2 \\ \bar{\theta}_1\theta_2 f_3 & \theta_1\bar{\theta}_2 f_4 & \theta_2 f_1 & \bar{\theta}_2 f_2 \\ -\bar{\theta}_1 f_3 & -\theta_1 f_4 & -f_1 & -f_2 \\ \theta_2 f_3 & \bar{\theta}_2 f_4 & \theta_1\theta_2 f_1 & \bar{\theta}_1\bar{\theta}_2 f_2 \end{pmatrix}$$

The matrix of components

$$\psi^{\alpha i} = \begin{pmatrix} \psi^{11} & \psi^{12} & \psi^{13} & \psi^{14} \\ \psi^{21} & \psi^{22} & \psi^{23} & \psi^{24} \\ \psi^{31} & \psi^{32} & \psi^{33} & \psi^{34} \\ \psi^{41} & \psi^{42} & \psi^{43} & \psi^{44} \end{pmatrix}$$

transforms into

$$\psi^{\alpha i} = \begin{pmatrix} -\psi^{33} & -\psi^{34} & -\psi^{31} & -\psi^{32} \\ \psi^{43} & \psi^{44} & \psi^{41} & \psi^{42} \\ -\psi^{13} & -\psi^{14} & -\psi^{11} & -\psi^{12} \\ \psi^{23} & \psi^{24} & \psi^{21} & \psi^{22} \end{pmatrix}$$

In general, under *space inversion*, the matrix of the spinor basis elements

$$\xi_{\alpha i} = \begin{pmatrix} f_1 & f_2 & \bar{\theta}_1 f_3 & \theta_1 f_4 \\ \theta_1\theta_2 f_1 & \bar{\theta}_1\bar{\theta}_2 f_2 & \theta_2 f_3 & \bar{\theta}_2 f_4 \\ \theta_1 f_1 & \bar{\theta}_1 f_2 & f_3 & f_4 \\ \theta_2 f_1 & \bar{\theta}_2 f_2 & \bar{\theta}_1\theta_2 f_3 & \theta_1\bar{\theta}_2 f_4 \end{pmatrix}$$

transforms into

$$\xi'_{\alpha i} = \begin{pmatrix} -f_3 & -f_4 & -\theta_1 f_1 & -\bar{\theta}_1 f_2 \\ \bar{\theta}_1\theta_2 f_3 & \theta_1\bar{\theta}_2 f_4 & \theta_2 f_1 & \bar{\theta}_2 f_2 \\ -\bar{\theta}_1 f_3 & -\theta_1 f_4 & -f_1 & -f_2 \\ \theta_2 f_3 & \bar{\theta}_2 f_4 & \theta_1\theta_2 f_1 & \bar{\theta}_1\bar{\theta}_2 f_2 \end{pmatrix}$$

The matrix of components

$$\psi^{\alpha i} = \begin{pmatrix} \psi^{11} & \psi^{12} & \psi^{13} & \psi^{14} \\ \psi^{21} & \psi^{22} & \psi^{23} & \psi^{24} \\ \psi^{31} & \psi^{32} & \psi^{33} & \psi^{34} \\ \psi^{41} & \psi^{42} & \psi^{43} & \psi^{44} \end{pmatrix}$$

transforms into

$$\psi'^{\alpha i} = \begin{pmatrix} -\psi^{33} & -\psi^{34} & -\psi^{31} & -\psi^{32} \\ \psi^{43} & \psi^{44} & \psi^{41} & \psi^{42} \\ -\psi^{13} & -\psi^{14} & -\psi^{11} & -\psi^{12} \\ \psi^{23} & \psi^{24} & \psi^{21} & \psi^{22} \end{pmatrix}$$

The spinor of the 1st ideal transforms into the spinor of the 3rd ideal

Generalized Dirac equation (Dirac-Kähler equation[1])

$$(i\gamma^\mu \partial_\mu - m)\Phi = 0$$

$\Phi = \phi^A \gamma_A = \psi^{\tilde{A}} \xi_{\tilde{A}} = \psi^{\alpha i} \xi_{\alpha i}$ ← Spinor basis of $Cl(1,3)$

α is spinor index of a left minimal ideal.
i runs over four left ideals of $Cl(1,3)$

$\langle (\xi^{\tilde{A}})^\ddagger \gamma^\mu \xi_{\tilde{B}} \rangle_S \equiv (\gamma^\mu)^{\tilde{A}}{}_{\tilde{B}}$

$$\left(i(\gamma^\mu)^{\tilde{A}}{}_{\tilde{B}} \partial_\mu - m\delta^{\tilde{A}}{}_{\tilde{B}} \right) \psi^{\tilde{B}} = 0$$

$(\gamma^\mu)^{\tilde{A}}{}_{\tilde{B}} = (\gamma^\mu)^\alpha{}_\beta \delta^i{}_j$

$$\left(i(\gamma^\mu)^\alpha{}_\beta \partial_\mu - m\delta^\alpha{}_\beta \right) \psi^{\beta i} = 0$$

$$(i\gamma^\mu \partial_\mu - m)\psi^i = 0$$

Here we omit spinor index α

Action

$$I = \int \mathrm{d}^4 x \, \bar{\psi}^i (i\gamma^\mu \partial_\mu - m)\psi^j z_{ij}$$

Metric in spinor space

$$(\xi_{\tilde{A}})^\ddagger * \xi_{\tilde{B}} = z_{\tilde{A}\tilde{B}} = z_{(\alpha i)(\beta j)} = z_{\alpha\beta} z_{ij}$$

$$z_{ij} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad z_{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

[1]E. Kähler, Rendiconti di Matematica 21 (1962) 425;
S.I. Kruglov, Dirack-Kähler Equation, arXiv: hep-th/0110251 (and many references therein)
D. Spehled, and G.C. Marques, Eur. Phys. J. 61 (2009) 75

Gauge invariant action:

$$I = \int \mathrm{d}^4 x \, \bar{\psi}^i (i \gamma^\mu \mathrm{D}_\mu - m) \psi^j z_{ij}$$

$$i, j = 1, 2, 3, 4$$

$$D_\mu \psi^i = \partial_\mu \psi^i + G_\mu{}^i{}_j \psi^j$$

This action contains the ordinary particles and mirror particles.

This index is omitted

$$\psi^{\alpha i} \equiv \psi^i = \begin{pmatrix} \psi^{11} & \psi^{12} & \psi^{13} & \psi^{14} \\ \psi^{21} & \psi^{22} & \psi^{23} & \psi^{24} \\ \psi^{31} & \psi^{32} & \psi^{33} & \psi^{34} \\ \psi^{41} & \psi^{42} & \psi^{43} & \psi^{44} \end{pmatrix}$$

The $\mathrm{SU}(2)$ gauge group acting within the 1st and 2nd ideal can be interpreted as the weak interaction gauge group for ordinary particles.

The $\mathrm{SU}(2)$ gauge group acting within the 3rd and 4th ideal can be interpreted as the weak interaction gauge group for mirror particles.

Gauge covariant action:

$$I = \int \mathrm{d}^4 x \, \bar{\psi}^i (i \gamma^\mu \mathrm{D}_\mu - m) \psi^j z_{ij}$$

$$i, j = 1, 2, 3, 4$$

$$D_\mu \psi^i = \partial_\mu \psi^i + G_\mu{}^i{}_j \psi^j$$

This action contains the ordinary particles and mirror particles.

This index is omitted

$$\psi^{\alpha i} \equiv \psi^i = \begin{pmatrix} \psi^{11} & \psi^{12} & \psi^{13} & \psi^{14} \\ \psi^{21} & \psi^{22} & \psi^{23} & \psi^{24} \\ \psi^{31} & \psi^{32} & \psi^{33} & \psi^{34} \\ \psi^{41} & \psi^{42} & \psi^{43} & \psi^{44} \end{pmatrix}$$

The $\mathrm{SU}(2)$ gauge group acting within the 1st and 2nd ideal can be ... the weak inte... group for orc...

The $\mathrm{SU}(2)$ gauge group acting within the 3rd and 4th ... erpreted as ... ction gauge ... r particles.

The corresponding two kinds of weak interaction gauge fields that can be transformed into each other by space inversion are contained in $G_\mu{}^i{}_j$.

Mirror particles were first proposed by Lee and Yang, Phys. Rev. 104 (1956) 254

Subsequently, the idea of mirror particles has been pursued by

I.Yu. Kobzarev, L.B. Okun, I.Ya. Pomeranchuk, Soviet J. Nucl. Phys. 5 (1966) 837.
M. Pavšič, Int. J. Theor. Phys. 9 (1974) 229.
E.W. Kolb, D. Seckel, M.S. Turner, Nature 314 (1985) 415
R. Foot, H. Lew, R.R. Volkas, Phys. Lett. B 272 (1991) 67;
R. Foot, H. Lew, R.R. Volkas, Mod. Phys. Lett. A 7 (1992) 2567;
R. Foot, Mod. Phys. Lett. 9 (1994) 169;
R. Foot, R.R. Volkas, Phys. Rev. D 52 (1995) 6595.

The possibility that mirror particles are responsible for dark matter has been explored in many works, e.g.:

H. M. Hodges, Phys. Rev. D 47 (1993) 456;
R. Foot, Phys. Lett. B 452 (1999) 83;
R. Foot, Phys. Lett. B 471 (1999) 191;
R.N. Mohapatra, Phys. Rev. D 62 (2000) 063506;
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Z. Berezhiani, D. Comelli, F. Villante, Phys. Lett. B 503 (2001).

A demonstration that mirror particles can be explained in terms of algebraic spinors (elements of Clifford algebras) was presented in

M. Pavšič, Phys. Lett. B 692 (2010) 212.

Clifford algebras and the concept of algebraic spinors opens Pandora's box of possibilities that have been explored in the attempts to find a unified theory of fundamental particles and forces. See, e.g.,

M. Pavšič: The Landscape of Theoretical Physics: A Global view; From Point Particles to the Brane World and Beyond, in Search of a Unifying Principle (Kluwer Academic, 2001)

Class. Quant. Grav. 20, 2697-2714 (2003); gr-qc/0111092

Kaluza-Klein theory without extra dimensions: Curved Clifford space, Phys. Lett. B614, 85-95 (2005); hep-th/0412255

Spin gauge theory of gravity in Clifford space: A Realization of Kaluza-Klein theory in 4- dimensional spacetime, Int. J. Mod. Phys. A21, 5905-5956 (2006); gr-qc/0507053

Beyond the relativistic point particle: A reciprocally invariant system, Phys. Lett. B 680, 526-532 (2009)

A Novel View on the Physical Origin of E8, J. Phys. A 41 (2008) 332001; 0806.4365 [hep-th]

On the relativity in configurations space: A renewed physics in sight, 0912.3669 [gr-qc]

Other authors: Hestenes, Smith, Trayling, Baylis, Roepstorff, Chisholm, Crawford, Castro, Schmeikel

Conclusion

Normally, our measuring tools (and reference frames) rotate as vectors: $v' = \mathrm{R}\, v\, \mathrm{R}^{-1}$.
Then any other Clifford number, Φ, also transforms as $\Phi' = \mathrm{R}\, \Phi\, \mathrm{R}^{-1}$.
This means that spinors, since being embedded in Φ, transform in the same way.

According to Piazzese, such behavior of spinors under rotations is an argument against spinors as members of minimal ideals of a Clifford algebra, because spinors must transform as $\Psi' = \mathrm{R}\, \Psi$.

But if we take into account the transformations within the entire Clifford algebra, then any Clifford number can transform as $\Phi' = \mathrm{R}\, \Phi\, \mathrm{S}$, and so can a spinor.
In particular, if $\mathrm{S} = 1$, then we have the usual transformation of spinors.

When our reference frame undergoes a space inversion, then a spinor of one minimal ideal transforms into a spinor of another minimal ideal of Clifford algebra.

We have pointed out how the enigmatic properties of weak interactions under space inversion can be understood in terms of geometric (algebraic) spinors.

Auxiliary slides

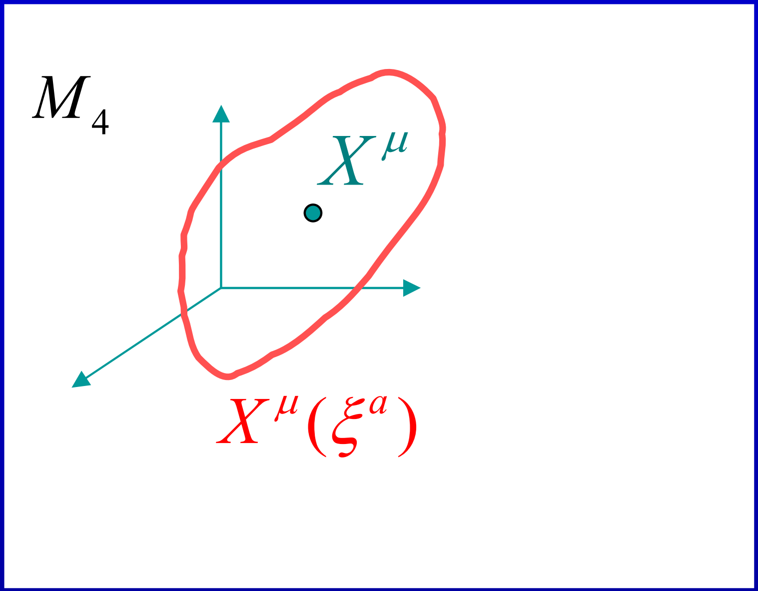
Not presented in this talk. Intended for discussion only.

Finite dimensional description of extended objects

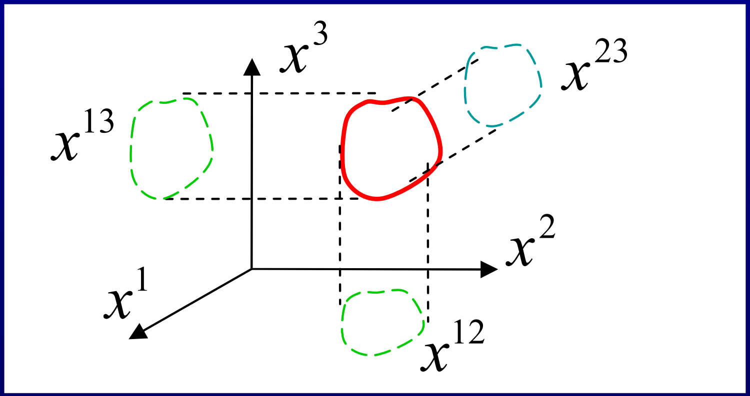
The Earth has a huge (practically infinite) number of degree of freedom. And yet, when describing the motion of the Earth around the Sun, we neglect them all, except for the coordinates of the centre of mass.

Instead of infinitely many degrees of freedom associated with an extended object, we may consider a finite number of degrees of freedom.

Strings and branes have infinitely many degrees of freedom.
But at first approximation we can consider just the centre of mass.

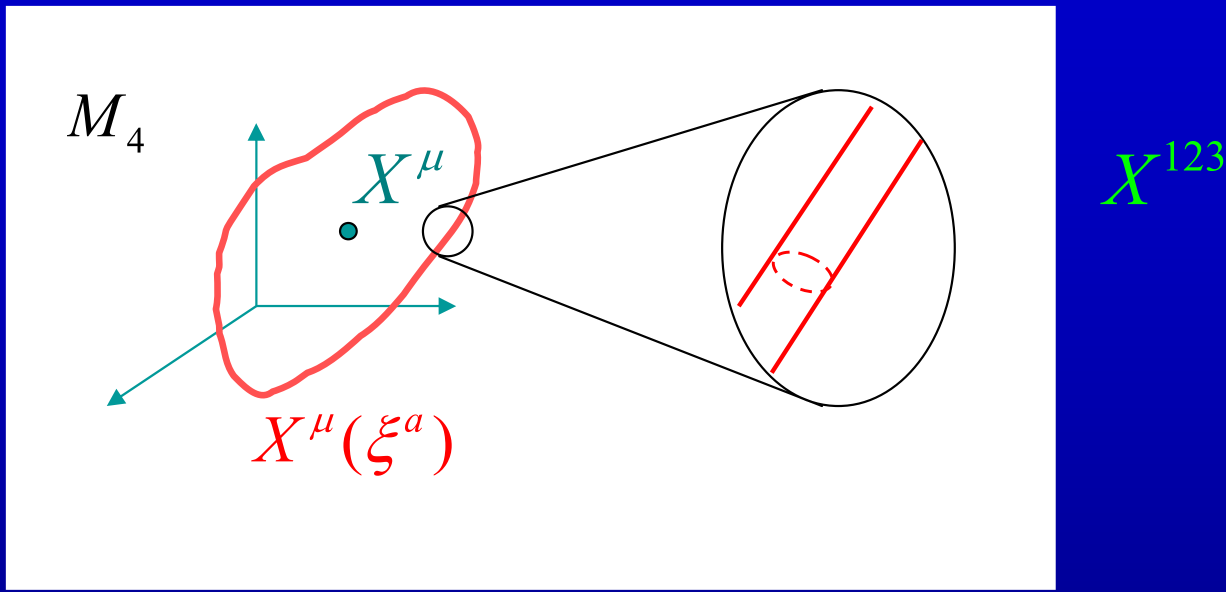


Next approximation is in considering the holographic coordinates of the oriented area enclosed by the string.



We may go further and search for eventual thickness of the object.

If the string has finite thickness, i.e., if actually it is not a string, but a 2-brane, then there exist the corresponding volume degrees of freedom.



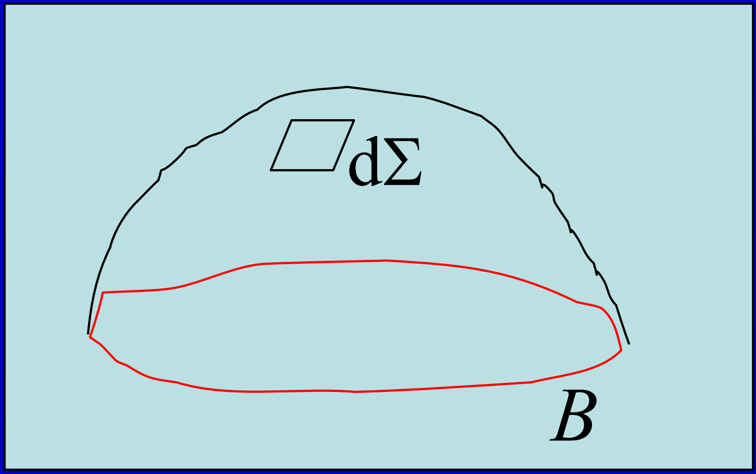
In general, for an extended object in M_4, we have 16 coordinates

$$x^M \equiv x^{\mu_1 \dots \mu_r}, \qquad r = 0,1,2,3,4$$

They are the projections of r-dimensional volumes (areas) onto the coordinate planes.

Oriented r-volumes can be elegantly described by Clifford algebra.

$$\mathrm{d}\Sigma = \mathrm{d}\xi_1 \wedge \mathrm{d}\xi_2 = \mathrm{d}\xi_1^a \, \mathrm{d}\xi_2^b \, e_a \wedge e_b = \frac{1}{2}\mathrm{d}\xi^{ab} \, e_a \wedge e_b$$



$$\mathrm{d}\xi^{ab} = \mathrm{d}\xi_1^a \, \mathrm{d}\xi_2^b - \mathrm{d}\xi_2^a \, \mathrm{d}\xi_1^b$$

$$e_a = \partial_a X^\mu \gamma_\mu$$

$X^{\mu\nu}$

$$\int_{\Sigma_B} \mathrm{d}\Sigma \equiv \frac{1}{2} X^{\mu\nu} \gamma_\mu \wedge \gamma_\nu = \frac{1}{2}\int_{\Sigma_B} \mathrm{d}\xi^{ab} \partial_a X^\mu \partial_b X^\nu \gamma_\mu \wedge \gamma_\nu$$

$$= \frac{1}{2}\int_{\Sigma_B} \mathrm{d}\xi^{ab} \frac{1}{2}(\partial_a X^\mu \partial_b X^\nu - p_a X^\nu \partial_b X^\mu) \gamma_\mu \wedge \gamma_\nu$$

$$X^{\mu\nu}[B] = \frac{1}{2}\int_{\Sigma_B} \mathrm{d}\xi^{ab} (\partial_a X^\mu \partial_b X^\nu - \partial_a X^\nu \partial_b X^\mu)$$

$$X^{\mu\nu}[B] = \tfrac{1}{2}\oint_B \mathrm{d}s \left(X^\mu \frac{\partial X^\nu}{\partial s} - X^\nu \frac{\partial X^\mu}{\partial s} \right)$$

Mapping :

$$X^\mu\left(\xi^a\right) \longrightarrow X^{\mu\nu}$$

Instead of the usual relativity formulated in spacetime in which the interval is

$$\mathrm{d}s^2 = \eta_{\mu\nu}\,\mathrm{d}x^\mu \mathrm{d}x^\nu$$

we are studying the theory in which the interval is extended to the space of r-volumes (called Clifford space):

$$\mathrm{d}S^2 = G_{MN}\,\mathrm{d}x^M \mathrm{d}x^N \qquad \mathrm{d}x^M \equiv \mathrm{d}x^{\mu_1 \ldots \mu_r}, \quad r = 0, 1, 2, 3, 4$$

Coordinates of Clifford space can be used to model extended objects. They are a generalization of the concept of center of mass.

Instead of describing extended objects in ``full detail'', we can describe them in terms of the center of mass, area and volume coordinates.

In particular, extended objects can be fundamental strings or branes.

Quadratic form in C-space

$$\mathrm{d}\,S^2 \equiv |\mathrm{d}X|^2 \equiv \mathrm{d}X^{\ddagger} * \mathrm{d}X = \mathrm{d}x^M \mathrm{d}x^N G_{MN} \equiv \mathrm{d}x^M \mathrm{d}x_M$$

where

$$\mathrm{d}X = \mathrm{d}x^M \gamma_M \equiv \mathrm{d}x^{\mu_1\mu_2\ldots\mu_r}\gamma_{\mu_1\mu_2\ldots\mu_r}, \qquad r = 0,1,2,3,4$$

Metric

$$G_{MN} = \gamma_M^{\ddagger} * \gamma_N \equiv \langle \gamma_M^{\ddagger}\,\gamma_N \rangle_0$$

Reversion

$$(\gamma_{\mu_1}\gamma_{\mu_2}\ldots\gamma_{\mu_r})^{\ddagger} = \gamma_{\mu_r}\ldots\gamma_{\mu_2}\gamma_{\mu_1}$$

Signature: $+ + + + + + + + - - - - - - - -$ (8,8)

In flat C-space:

$$\gamma_{\mu_1\mu_2\ldots\mu_r} = \gamma_{\mu_1} \wedge \gamma_{\mu_2} \wedge \ldots \wedge \gamma_{\mu_r}$$

at every point $\mathcal{E} \in C$

Dynamics

Action:

$$I = \int d\tau \, (\eta_{MN} \dot{X}^M \dot{X}^N)^{1/2}$$

Generalization of ordinary relativity

Equations of motion:

$$\ddot{X}^M \equiv \frac{\mathrm{d}^2 X^M}{\mathrm{d}\tau^2} = 0$$

These equations imply area (volume) motion

Metric: η_{MN}

Diagonal metric

Signature: $+ + + + + + + + - - - - - - - -$ (8,8)

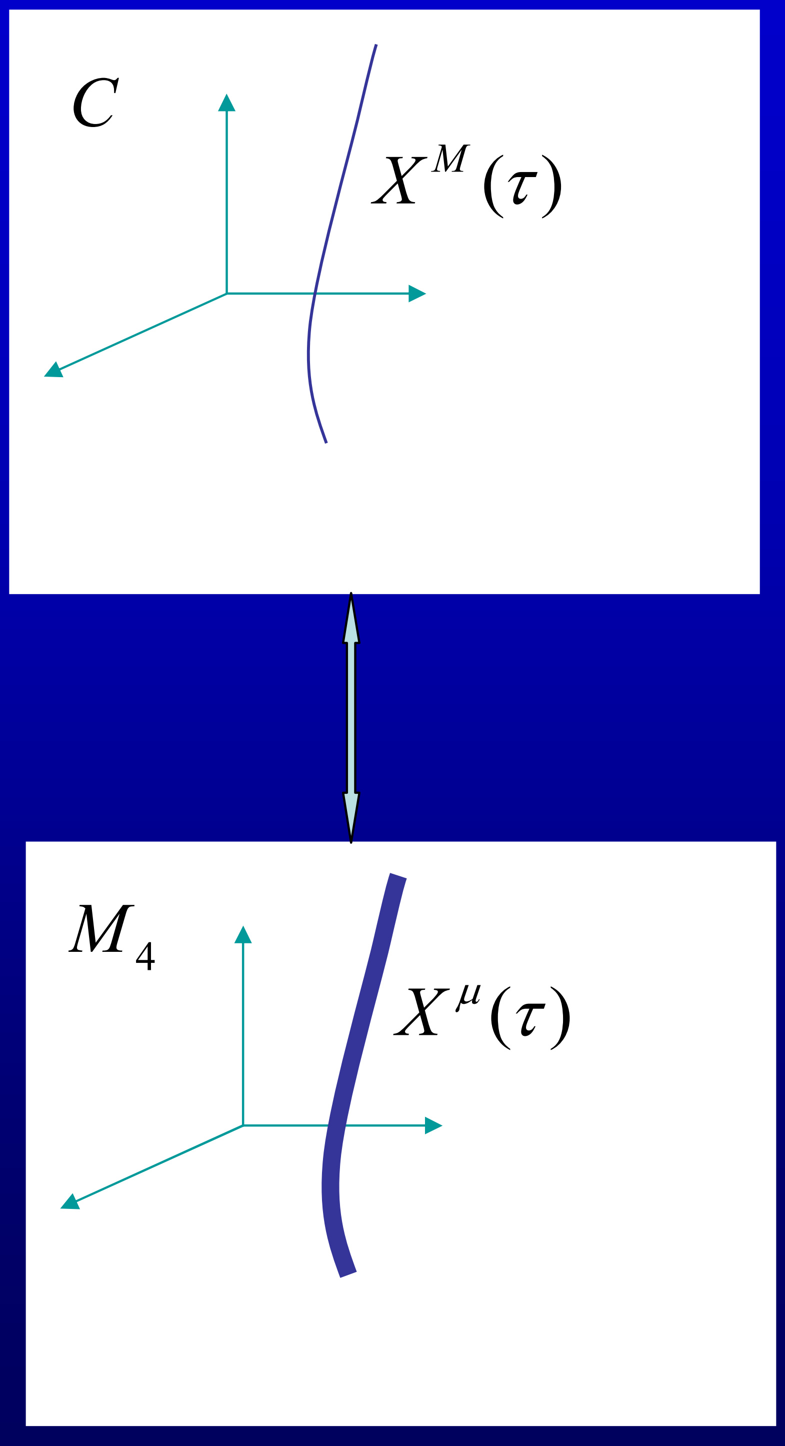
The above dynamics holds for tensionless branes.
For the branes with tension one has to introduce
curved Clifford space.

Thick point particles and strings

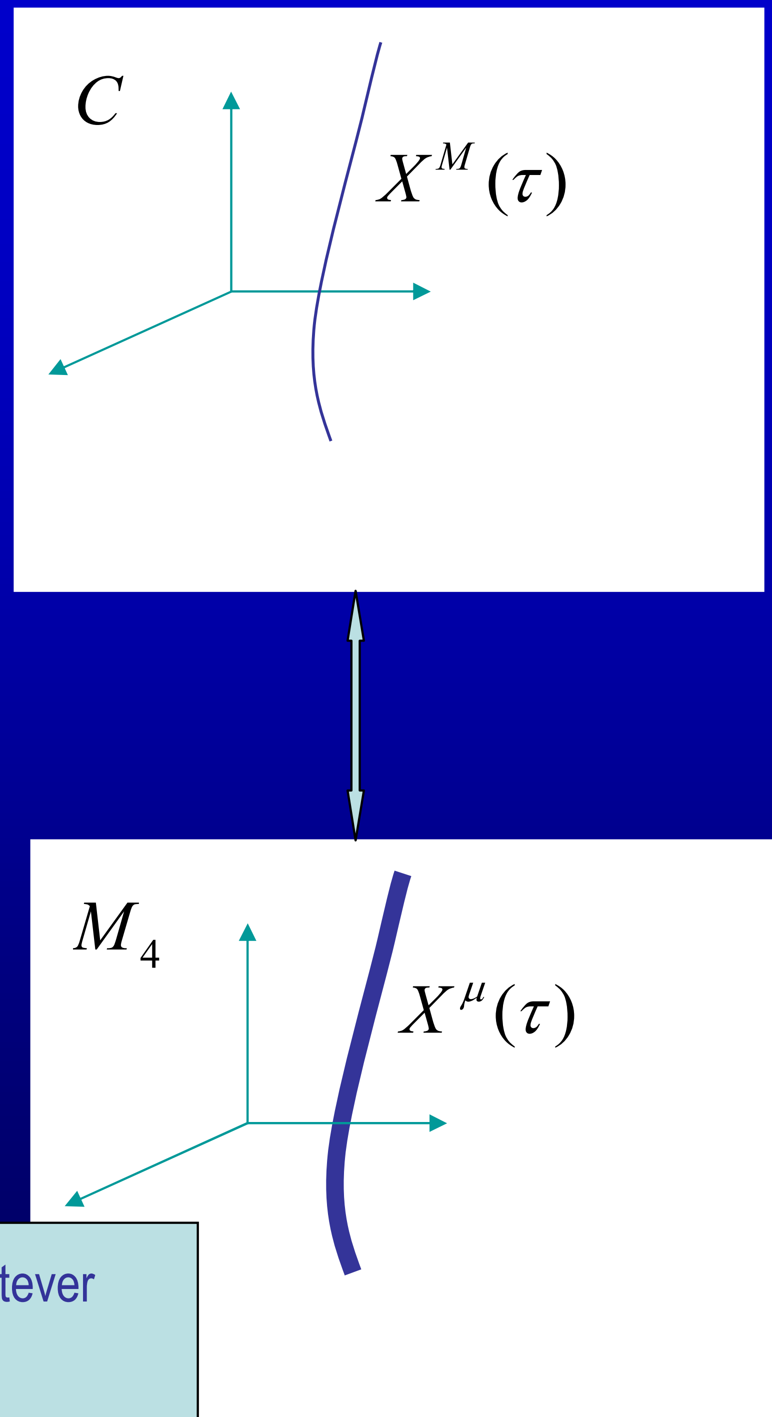
A world line in C represents the evolution of a `thick' particle in spacetime M_4

Thick particle can be an aggregate p-branes for various p=0,1,2,…

But such interpretation is not obligatory.



A world line in C represents the evolution of a `thick' particle in spacetime M_4

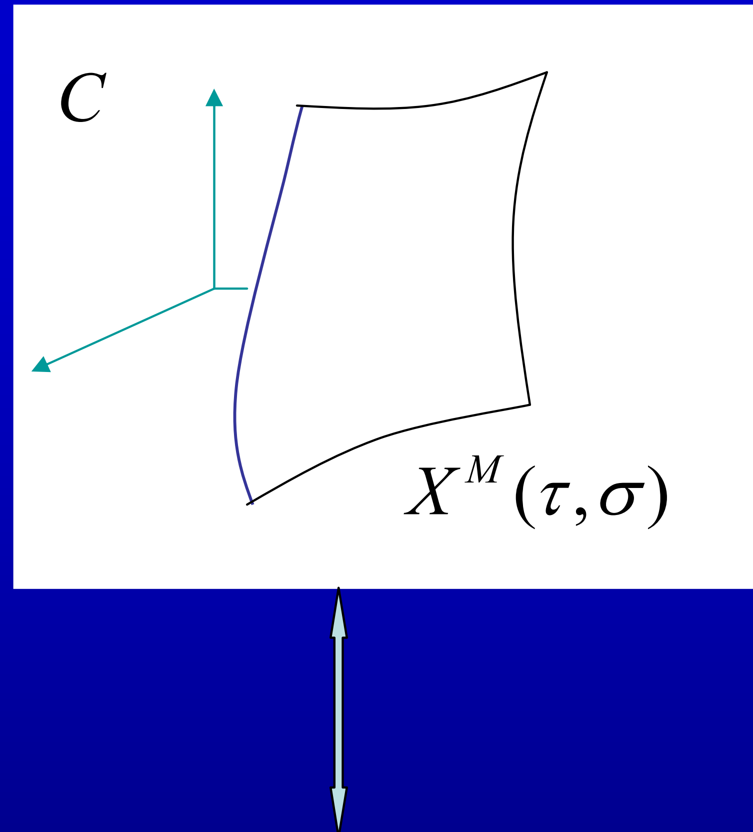


Thick particle can be an aggregate *p*-branes for various *p*=0,1,2,…

But such interpretation is not obligatory.

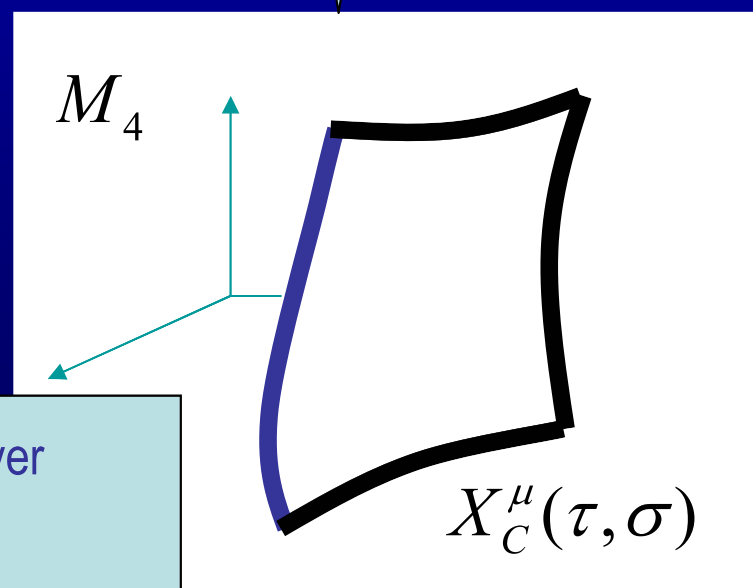
Thick particle may be a conglomerate of whatever extended objects that can be sampled by polyvector coordinates $X^M \equiv X^{\mu_1 \mu_2 \dots \mu_r}$

A world sheet in *C* represents the evolution of a `thick' string in spacetime M_4



Thick string can be an aggregate *p*-branes for various *p*=0,1,2,…

But such interpretation is not obligatory.



Thick string may be a conglomerate of whatever extended objects that can be sampled by polyvector coordinates $X^M \equiv X^{\mu_1\mu_2\ldots\mu_r}$

$X^{\mu}(\tau)$

Usual strings are infinitely thin object. Although called `extended objects', they are not fully extended.

Instead of infinitely thin strings we thus consider thick strings. Their thickness is encoded in polyvector coordinates $X^M \equiv X^{\mu_1 \mu_2 ... \mu_r}$.

Infinitely thin strings are singular objects

String action

$$I = \frac{\kappa}{2} \int \mathrm{d}\tau \, \mathrm{d}\sigma \, (\dot{X}^M \dot{X}^N - X'^M X'^N) \, G_{MN}$$

Conformal gauge

No extra dimensions of the spacetime are required

The necessary extra dimensions for consistency of string theory are in 16-dimensional Clifford space.

Jackiw-Kim-Noz definition of vacuum

No central terms in the Virasoro algebra, if the space in which the string lives has signature $(+ + + \; ... \; - - -)$

The space in which out string lives is Clifford space.
Its dimension is 16, and signature (8,8).

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$$X^M = (x, x^\mu, x^{\mu\nu}, \ldots)$$

$$\gamma^M = (\underline{1}, \gamma_\mu, \gamma_{\mu\nu}, \ldots)$$

$X^M \gamma_M$ Polyvector

(It contains spinors)