Dark Energy Modified Gravity: from non-singular universe to Little Rip cosmology

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Contents:

Little Rip and other non-singular universes

Spatially flat Friedman-Lemaitre-Robertson-Walker (FLRW) universe is given by the metric

$$ds^2 = -dt^2 + a(t)^2 \sum_{i=1,2,3} \left(dx^i\right)^2 . \tag{1}$$

The FLRW equations in the Einstein gravity coupled with perfect fluid are well-known to be:

$$\rho = \frac{3}{\kappa^2} H^2 , \quad p = -\frac{1}{\kappa^2} \left(3H^2 + 2\dot{H} \right) . \tag{2}$$

Here the Hubble rate H is defined by $H \equiv \dot{a}/a$. The equation of state (EoS) parameter w is defined by the ratio of the pressure p and the energy density $w \equiv p/\rho$.

LATE-TIME COSMIC ACCELERATION. DARK ENERGY AS COSMIC FLUID

The future of the universe is mainly governed by the equation of state (EoS) parameter $w_{\rm DE}$ of the dark energy $w_{\rm DE} \equiv p_{\rm DE}/\rho_{\rm DE}$. The accelerating expansion can be generated if $w_{\rm DE} < -1/3$. The observational data indicate that $w_{\rm DE}$ is close to -1. If the $w_{\rm DE}$ is exactly -1, the present universe is described by the ΛCDM model, where the cosmological term generates the accelerating expansion and the universe evolves to the asymptotic de Sitter space-time. The dark energy with $-1 < w_{\rm DE} < -1/3$ is called as quintessence and that with $w_{\rm DE} < -1$ as phantom. Even if the dark energy is quintessence, soft singularity might be generated in the finite-time future.

FINITE-TIME FUTURE SINGULARITIES

The classification of the finite-time future singularities is given in S. Nojiri, S.D.O. and S. Tsujikawa, hep-th/0501025, as follows:

- Type I ("Big Rip") : For $t \to t_s$, $a \to \infty$, $\rho_{\rm DE} \to \infty$ and $|p_{\rm DE}| \to \infty$. This also includes the case of $\rho_{\rm DE}$, $p_{\rm DE}$ being finite at t_s.
- Type II ("sudden"): For $t \to t_s$, $a \to a_s$, $\rho_{\rm DE} \to \rho_s$ and $|p_{\rm DE}| \to \infty$.
- Type III : For $t \to t_s$, $a \to a_s$, $\rho_{\rm DE} \to \infty$ and $|p_{\rm DE}| \to \infty$.
- Type IV : For $t \to t_s$, $a \to a_s$, $\rho_{\rm DE} \to 0$, $|p_{\rm DE}| \to 0$ and higher derivatives of H diverge. This also includes the case in which $p_{\rm DE}$ ($\rho_{\rm DE}$) or both of $p_{\rm DE}$ and $\rho_{\rm DE}$ tend to some finite values, whereas higher derivatives of H diverge.

It is very interesting to understand if the solution of singularity problem may be found within fluid dark energy.
Let us concentrate now on phantom era which normally leads to Big Rip singularity in finite future if $w < -1$. Even if $w < -1$, if w approaches to -1 sufficiently rapidly, the singularity is not always generated. First of all, transient phantom era is possible. Moreover, one construct the phantom models where w asymptotically tends to -1 so that the universe ends up in asymptotically de Sitter space.
Recently, new scenario to avoid future singularity has been proposed in P.Frampton, K. Ludwick and R. Scherrer, arXiv:1106.4996. Since $w(< -1) \to -1$ asymptotically, the finite-time singularity is avoided and the singularity is shifted to infinite future. Even in such non-singular cosmology, if H goes to infinity when $t \to \infty$, it was found that there might occur the dissolution of bound objects sometime in future, similarly to Big Rip singularity. That is why the scenario was called Little Rip cosmology. The phantom scalar models to describe Little Rip were introduced in P. Frampton, K. Ludwick, S. Nojiri, S.D.O. and R. Scherre, arXiv:1108.0067. Finally, there is possibility of pseudo-rip scenario which lies between Little Rip and cosmological constant cosmology with Hubble rate tending to constant at infinite future.

LITTLE RIP COSMOLOGY AND OTHER NON-SINGULAR UNIVERSES

As the universe expands, the relative acceleration between two points separated by a distance l is given by $l \cdot \ddot{a}/a$, where a is the scale factor. If there is a particle with mass m at each of the points, an observer at one of the masses will measure an inertial force on the other mass of

$$F_{\rm iner} = m \cdot l \cdot \ddot{a}/a = m \cdot l \left(\dot{H} + H^2 \right) . \tag{3}$$

Let us assume the the two particles are bound by a constant force F_0. If $F_{\rm iner}$ is positive and greater than F_0, the two particles become unbound. This is the "rip" produced by the accelerating expansion. Note that equation (3) shows that a rip always occurs when either H diverges or $\dot{H}$ diverges (assuming $\dot{H} > 0$). The first case corresponds to a "Big Rip", while if H is finite, but $\dot{H}$ diverges with $\dot{H} > 0$, we have a Type II or "sudden future" singularity [Barrow, 2004], which also leads to a rip. Even if H or $\dot{H}$ goes to infinity at the infinite future, the inertial force becomes larger and larger, and any bound object is ripped, which is called "Little Rip".

The acceleration by the gravitational force between the sun and the earth is given by $a_{\rm g} = l \cdot \omega_A^2$. Here l is the distance between the sun and the earth and ω_A is the angular speed $\omega_A = 2\pi/\left(1\,{\rm year}\right) = 1.99 \times 10^{-6}\,{\rm s}^{-1}$. If the acceleration $a_{\rm e}$ of the inertial force by the expansion (3) exceeds $a_{\rm g}$, there occurs the rip between the earth and the sun, that is,

$$a_{\rm e} = l\left(\dot H + H^2\right) \sim lH^2 > a_{\rm g}\,. \tag{4}$$

A MODEL OF LITTLE RIP COSMOLOGY

Ansatz:

$$H = H_0^{(\mathrm{I})} \mathrm{e}^{\lambda t} \,, \tag{5}$$

Here $H_0^{(\mathrm{I})}$ and λ are positive constants. Eq. (5) shows that there is no curvature singularity for finite t.
If $\dot{H} > 0$, $w < -1$:

$$w = -1 - \frac{2\lambda}{3H_0^{(\mathrm{I})}} \mathrm{e}^{-\lambda t} \,, \tag{6}$$

and therefore $w < -1$ and $w \to -1$ when $t \to +\infty$, and w is always less than -1 when $\dot{H}$ is positive. The parameter $2\lambda/\sqrt{3}$ is bounded as

$$2.37 \times 10^{-3}\,\mathrm{Gyr}^{-1} \leq \lambda \leq 8.37 \times 10^{-3}\,\mathrm{Gyr}^{-1} \,, \tag{7}$$

by the results of the Supernova Cosmology Project Amanullah et al, 2010.
H is always finite but increases exponentially, what generates the strong inertial force. The inertial force becomes larger and larger and any bound object is ripped.

ASYMPTOTICALLY DE SITTER PHANTOM MODEL

$$H = H_0^{(\mathrm{II})} - H_1^{(\mathrm{II})} \mathrm{e}^{-\lambda t} \,. \tag{8}$$

Here $H_0^{(\mathrm{II})} > H_1^{(\mathrm{II})}$ and $t > 0$. Since the second term decreases when t increases, the universe goes to asymptotically de Sitter space-time. Then

$$w = -1 - \frac{2\lambda H_1^{(\mathrm{II})} \mathrm{e}^{-\lambda t}}{3 \left(H_0^{(\mathrm{II})} - H_1^{(\mathrm{II})} \mathrm{e}^{-\lambda t} \right)^2} \,. \tag{9}$$

As in the previous example (5), $w < -1$ and $w \to -1$ when $t \to +\infty$. In this model, there does not occur the Little Rip. The inertial force in (3) generated by the expansion of the universe is finite since the magnitudes of H and $\dot{H}$ are bounded in the model (8).

ASYMPTOTICALLY DE SITTER QUINTESSENCE DARK ENERGY

Let us consider the quintessence model where $w > -1$ but $w \to -1$ when $t \to +\infty$ as follows,

$$H = H_0^{(\mathrm{III})} + H_1^{(\mathrm{III})} \mathrm{e}^{-\lambda t} \,. \tag{10}$$

Here $H_0^{(\mathrm{III})} > H_1^{(\mathrm{III})}$ and $t > 0$. Since the second term decreases when t increases, the universe goes to asymptotically de Sitter space-time. Here

$$w = -1 + \frac{2\lambda H_1^{(\mathrm{III})} \mathrm{e}^{-\lambda t}}{3\left(H_0^{(\mathrm{III})} - H_1^{(\mathrm{III})} \mathrm{e}^{-\lambda t}\right)^2} \,. \tag{11}$$

Hence, the EoS parameter is always larger than $-1$and $w \to -1$ when $t \to +\infty$. Therefore the universe is in non-phantom phase.

A MODEL UNIFYING INFLATION WITH LITTLE RIP DARK ENERGY ERA

As one more example, we consider the realistic model which contains the inflation at $t \to -\infty$, phantom crossing at $t = 0$, and the Little Rip when $t \to \infty$:

$$H = H_0^{(\mathrm{IV})} \cosh \lambda t \,. \tag{12}$$

Since $\dot{H} = H_0^{(\mathrm{IV})}(\lambda) \sinh \lambda t$, we find $\dot{H} < 0$ when $t < 0$, that is, the universe is in non-phantom phase and $\dot{H} > 0$ when $t > 0$, that is, the universe is in phantom phase. There occurs the phantom crossing at $t = 0$. Therefore the present universe corresponds to $t \sim 0$. When $\lambda t \gg 1$, we find that the Hubble rate H behaves as $H \sim \frac{H_0^{(\mathrm{IV})}}{2} \mathrm{e}^{\lambda t}$ and therefore there occurs the Little Rip. The EoS parameter w is

$$w = -1 - \frac{2\lambda \sinh \lambda t}{2H_0^{(\mathrm{IV})} \cosh^2 \lambda t} \,. \tag{13}$$

Hence, $w < -1$ when $t > 0$ and $w > -1$ when $t < 0$. In the limit $t \to \pm\infty$, $w \to -1$. Thus when $t \to -\infty$, there occurs the accelerating expansion, which may correspond to the inflation in the early universe.

When $w = -\frac{1}{3}$, that is,

$$\frac{\lambda \sinh \lambda t}{H_0^{(\mathrm{IV})} \cosh^2 \lambda t} = -1 \,, \tag{14}$$

there occurs the transition between non-accelerating expansion and accelerating transition. For details of above scenario and its two scalar presentation, see Yu.Ito,S. Nojiri, S.D.O., arXiv:1111.5389.

Advantages

1. Modified gravity provides the very natural gravitational alternative for dark energy.
2. Modified gravity presents very natural unification of the early-time inflation and late-time acceleration.
3. It may serve as the basis for unified explanation of dark energy and dark matter.
4. Assuming that universe is entering the phantom phase, modified gravity may naturally describe the transition from non-phantom phase to phantom one without necessity to introduce the exotic matter.
5. Modified gravity quite naturally describes the transition from decceleration to acceleration in the universe evolution.
6. The effective dark energy dominance may be assisted by the modification of gravity.
7. Modified gravity is expected to be useful in high energy physics.
8. Despite quite stringent constraints from Solar System tests, there are versions of modified gravity which may be viable theories competing with General Relativity at current epoch.

I. Class of viable modified $f(R)$ gravities describing inflation and the onset of accelerated expansion

Nojiri, SDO, arXiv:0707.1941, 0710.1738; Cognola, Elizalde, Nojiri, SDO, Sebastiani, Zerbini, PRD77:046009,2008;PRD83:086006,2011

Starting action:

$$S = \frac{1}{\kappa^2} \int d^4 x \sqrt{-g} \left[R + f(R) \right] + S_{(m)} \tag{15}$$

Here $f(R)$ is a suitable function, which defines the modified gravitational part of the model. The general equation of motion in $F(R) \equiv R + f(R)$ gravity with matter is given by

$$\frac{1}{2} g_{\mu\nu} F(R) - R_{\mu\nu} F'(R) - g_{\mu\nu} \Box F'(R) + \nabla_\mu \nabla_\nu F'(R) = -\frac{\kappa^2}{2} T_{(m)\mu\nu} \ . \tag{16}$$

Elizalde, Nojiri, SDO, Sebastiani and Zerbini, arXiv:1012.2280.

Viable conditions in $F(R)$-gravity
In order to avoid anti-gravity effects, it is required that $F'(R) > 0$, namely, the positivity of the effective gravitational coupling.

Existence of a matter era and stability of cosmological perturbations.
On the critical points, $\dot{F}'(R) = 0$ and

$$\rho_{\rm eff} = \frac{1}{F'(R)} \left\{ \rho + \frac{1}{2\kappa^2} \left[(F'(R)R - F(R)) \right] \right\} , \tag{17}$$

$$p_{\rm eff} = \frac{1}{F'(R)} \left\{ p + \frac{1}{2\kappa^2} \left[-(F'(R)R - F(R)) \right] \right\} . \tag{18}$$

During matter era, $p_{\rm eff} \simeq 0$ and $\rho_{\rm eff} \simeq \rho / F'(R)$. As a consequence,

$$\frac{RF'(R)}{F(R)} = 1 \,, \qquad \frac{d}{dR} \left(\frac{RF'(R)}{F(R)} \right) = 0 \,. \tag{19}$$

This leads to

$$\frac{F''(R)}{F'(R)} = 0 \Rightarrow F''(R) \simeq 0 \,. \tag{20}$$

Since if $F''(R) < 0$ the theory is strongly unstable, $F''(R) \simeq 0^+$.

Local tests

The typical value of the curvature in the Solar System far from sources is $R = R^*$, where $R^* \simeq 10^{-61}$eV^2. If a Schwarzshild-de Sitter solution exists, it will be stable provided

$$\frac{F'(R^*)}{R^* F''(R^*)} > 1\,. \tag{21}$$

The stability of the solution is necessary in order to find the post-Newtonian parameters as in GR.

Exponential gravity

$$F(R) = R - 2\Lambda\left(1 - \mathrm{e}^{-R/R_0}\right)\,. \tag{22}$$

Here, $\Lambda \simeq 10^{-66}$eV^2 is the cosmological constant and $R_0 \simeq \Lambda$ a curvature parameter. In flat space one has $F(0) = 0$ and recovers the Minkowski solution. The model satisfied all viable conditions and it is consistent with the results of ΛCDM Model.

Inflation

A quite natural possibility is

$$F(R) = R - 2\Lambda\left(1 - \mathrm{e}^{-\frac{R}{R_0}}\right) - \Lambda_i\left(1 - \mathrm{e}^{-\left(\frac{R}{R_i}\right)^n}\right) + \gamma R^\alpha\,. \qquad (23)$$

This is the function discussed above with another one-step function reproducing the cosmological constant during inflation AND a power term necessary to obtain the exit from inflation ($\gamma \simeq 1/(4\Lambda_i)^{\alpha-1}$).
By taking into account all the viability conditions, the simplest choice of parameters to introduce in the function of Eq. (23) is

$$n = 4\,, \qquad \alpha = \frac{5}{2}\,, \qquad (24)$$

while the curvature R_i is set as

$$R_i = 2\Lambda_i\,. \qquad (25)$$

In this way, since $n > \alpha > 1$, we avoid the contribute of inflation and undesirable instability effects in the small-curvature regime. No anti-gravity effects. The unstable de Sitter solution describing inflation is

$$R_{\mathrm{dS}} = \frac{2\Lambda_i}{3-\alpha} \equiv 4\Lambda_i\,. \qquad (26)$$

Dark energy evolution

We will now be interested in the cosmological evolution of the dark energy density $\rho_{\rm DE} = \rho_{\rm eff} - \rho/F'(R)$ in the case of the two-step model of Eq. (23), near the late-time acceleration era.

We use the variable

$$y_{\rm H} \equiv \frac{\rho_{\rm DE}}{\rho_m^{(0)}} = \frac{H^2}{\tilde{m}^2} - a^{-3} - \chi a^{-4}\,. \quad (27)$$

Here, $\rho_m^{(0)}$ is the energy density of matter at present time, $\tilde{m}^2$ is the mass scale

$$\tilde{m}^2 \equiv \frac{\kappa^2 \rho_m^{(0)}}{3} \simeq 1.5 \times 10^{-67} \text{eV}^2\,, \quad (28)$$

and χ is defined as

$$\chi \equiv \frac{\rho_r^{(0)}}{\rho_m^{(0)}} \simeq 3.1 \times 10^{-4}\,, \quad (29)$$

where $\rho_r^{(0)}$ is the energy density of current radiation.

The EoS-parameter $\omega_{\rm DE}$ for dark energy is

$$\omega_{\rm DE} = -1 - \frac{1}{3}\frac{1}{y_{\rm H}}\frac{dy_{\rm H}}{d(\ln a)}\,. \quad (30)$$

By combining the Equations of motion of modified gravity theories, one gets

$$\frac{d^2 y_{\rm H}}{d(\ln a)^2} + J_1 \frac{d y_{\rm H}}{d(\ln a)} + J_2 y_{\rm H} + J_3 = 0\,, \tag{31}$$

where

$$J_1 = 4 + \frac{1}{y_{\rm H} + a^{-3} + \chi a^{-4}} \frac{1 - F'(R)}{6\tilde{m}^2 F''(R)}\,,$$

$$J_2 = \frac{1}{y_{\rm H} + a^{-3} + \chi a^{-4}} \frac{2 - F'(R)}{3\tilde{m}^2 F''(R)}\,,$$

$$J_3 = -3a^{-3} - \frac{(1 - F'(R))(a^{-3} + 2\chi a^{-4}) + (R - F(R))/(3\tilde{m}^2)}{y_{\rm H} + a^{-3} + \chi a^{-4}} \frac{1}{6\tilde{m}^2 F''(R)}\,,$$

and thus, we have

$$R = 3\tilde{m}^2 \left(\frac{d y_{\rm H}}{d \ln a} + 4 y_{\rm H} + a^{-3} \right)\,. \tag{32}$$

We will study our model,

$$F(R) = R - 2\Lambda\left(1 - \mathrm{e}^{-\frac{R}{R_0}}\right) - \Lambda_i\left(1 - \mathrm{e}^{-\left(\frac{R}{R_i}\right)^n}\right) + \gamma R^\alpha\,. \qquad (33)$$

The parameters of Eq. (33) are chosen as follows:

$$\begin{aligned} &\Lambda = (7.93)\tilde{m}^2\,, \\ &\Lambda_i = 10^{100}\Lambda\,, \\ &R_i = 2\Lambda_i\,, \qquad n = 4\,, \\ &\alpha = \frac{5}{2}\,, \qquad \gamma = \frac{1}{(4\Lambda_i)^{\alpha-1}}\,, \\ &R_0 = 0.6\Lambda\,, \quad 0.8\Lambda\,, \quad \Lambda\,. \end{aligned} \qquad (34)$$

Eq. (31) can be solved in a numerical way, in the range of $R_0 \ll R \ll R_i$ (matter era/current acceleration). $y_{\rm H}$ is then found as a function of the red shift z,

$$z = \frac{1}{a} - 1\,. \tag{35}$$

In solving Eq. (31) numerically we have taken the following initial conditions at $z = z_i$

$$\frac{dy_{\rm H}}{d(z)}\Big|_{z_i} = 0\,,$$
$$y_{\rm H}\Big|_{z_i} = \frac{\Lambda}{3\tilde{m}^2}\,, \tag{36}$$

which correspond to the ones of the ΛCDM model. This choice obeys to the fact that in the high red shift regime the exponential model is very close to the ΛCDM Model. The values of z_i have been chosen so that $RF''(z = z_i) \sim 10^{-7}$, assuming $R = 3\tilde{m}^2(z+1)^3 + 4\Lambda$. We have $z_i = 1.5$, 2.2, 2.5 for $R_0 = 0.6\Lambda$, 0.8Λ, Λ, respectively. In setting the parameters, we have used the last results of the WMAP, BAO and SN surveys (*Komatsu et al. [WMAP Collaboration], arXiv:0803.0547*).

We can also extrapolate the behavior of the density parameter of dark energy, $\Omega_{\rm DE}$,

$$\Omega_{\rm DE} \equiv \frac{\rho_{\rm DE}}{\rho_{\rm eff}} = \frac{y_{\rm H}}{y_{\rm H} + (z+1)^3 + \chi\,(z+1)^4}\,. \tag{37}$$

The data we have found are in accordance with the last and very accurate observations of our present universe, where:

$$\omega_{\rm DE} = -0.972^{+0.061}_{-0.060}\,,$$

$$\Omega_{\rm DE} = 0.721 \pm 0.015\,. \tag{38}$$

At the redshift $z = 0$ we obtain $\omega_{DE} = -0.994, -0.975, -0.950$ and $\Omega_{DE} = 0.726, 0.728, 0.732$ in the cases of $R_0 = 0.6\Lambda, 0.8\Lambda, \Lambda$, respectively.
The de Sitter solution is a final attractor of our system and describes an eternal accelerating expansion.

II. Cosmological reconstruction of modified F(R) gravity

Nojiri-SDO-Saez-Gomez, arXiv:0908.1269, Nojiri-SDO, hep-th/0611071, hep-th/0608008

Let us demonstrate that any FRW cosmology may be realized in specific $F(R)$ gravity.

The starting action of the $F(R)$ gravity is given by

$$S=\int d^4x\sqrt{-g}\left(\frac{F(R)}{2\kappa^2}+\mathcal{L}_{\rm matter}\right)\ . \quad (39)$$

The field equation corresponding to the first FRW equation is:

$$0=-\frac{F(R)}{2}+3\left(H^2+\dot H\right)F'(R)-18(\left(4H^2\dot H+H\ddot H\right)F''(R)+\kappa^2\rho\ . \quad (40)$$

We now rewrite Eq.(40) by using a new variable (which is often called e-folding) instead of the cosmological time t, $N = \ln \frac{a}{a_0}$. The variable N is related with the redshift z by $\mathrm{e}^{-N} = \frac{a_0}{a} = 1 + z$.
Since $\frac{d}{dt} = H \frac{d}{dN}$ and therefore $\frac{d^2}{dt^2} = H^2 \frac{d^2}{dN^2} + H \frac{dH}{dN} \frac{d}{dN}$,
one can rewrite (40) by

$$\begin{aligned} 0 &= -\frac{F(R)}{2} + 3\left(H^2 + HH'\right) F'(R) \\ &\quad -18(\left(4H^3 H' + H^2 \left(H'\right)^2 + H^3 H''\right) F''(R) + \kappa^2 \rho \ . \end{aligned} \quad (41)$$

Here $H' \equiv dH/dN$ and $H'' \equiv d^2 H/dN^2$.

If the matter energy density ρ is given by a sum of the fluid densities with constant EoS parameter w_i,

$$\rho = \sum_i \rho_{i0} a^{-3(1+w_i)} = \sum_i \rho_{i0} a_0^{-3(1+w_i)} \mathrm{e}^{-3(1+w_i)N} \ . \qquad (42)$$

Let the Hubble rate is given in terms of N via the function $g(N)$ as

$$H = g(N) = g\left(-\ln\left(1+z\right)\right) \ . \qquad (43)$$

Then scalar curvature takes the form: $R = 6g'(N)g(N) + 12g(N)^2$, which could be solved with respect to N as $N = N(R)$.

Then by using (42) and (43), one can rewrite (41) as

$$\begin{aligned} 0 \;=\; & -18\Big(4g\left(N\left(R\right)\right)^{3}g'\left(N\left(R\right)\right) \\ & +g\left(N\left(R\right)\right)^{2}g'\left(N\left(R\right)\right)^{2}+g\left(N\left(R\right)\right)^{3}g''\left(N\left(R\right)\right)\Big)\frac{d^{2}F(R)}{dR^{2}} \\ & +3\Big(g\left(N\left(R\right)\right)^{2}+g'\left(N\left(R\right)\right)g\left(N\left(R\right)\right)\Big)\frac{dF(R)}{dR}-\frac{F(R)}{2} \\ & +\sum_{i}\rho_{i0}a_{0}^{-3(1+w_{i})}\mathrm{e}^{-3(1+w_{i})N(R)}\;, \end{aligned} \tag{44}$$

which constitutes a differential equation for $F(R)$, where the variable is scalar curvature R.

Instead of g, if we use $G(N) \equiv g(N)^2 = H^2$, the expression (44) could be a little bit simplified:

$$\begin{aligned} 0 \quad = \quad & -9G(N(R))\left(4G'(N(R)) + G''(N(R))\right)\frac{d^2F(R)}{dR^2} \\ & + \left(3G(N(R)) + \frac{3}{2}G'(N(R))\right)\frac{dF(R)}{dR} \\ & - \frac{F(R)}{2} + \sum_i \rho_{i0} a_0^{-3(1+w_i)} \mathrm{e}^{-3(1+w_i)N(R)} \ . \end{aligned} \tag{45}$$

Note that the scalar curvature is given by $R = 3G'(N) + 12G(N)$.
Hence, when we find $F(R)$ satisfying the differential equation (44) or (45), such $F(R)$ theory admits the solution (43). Hence, such $F(R)$ gravity realizes above cosmological solution.

As an example, we reconstruct the $F(R)$ gravity which reproduces the ΛCDM-era but without real matter.
In the Einstein gravity, the FRW equation for the ΛCDM cosmology is given by

$$\frac{3}{\kappa^2}H^2 = \frac{3}{\kappa^2}H_0^2 + \rho_0 a^{-3} = \frac{3}{\kappa^2}H_0^2 + \rho_0 a_0^{-3}\mathrm{e}^{-3N} \ . \tag{46}$$

Here H_0 and ρ_0 are constants.
The (effective) cosmological constant Λ in the present universe is given by $\Lambda = 12H_0^2$. Then one gets

$$G(N) = H_0^2 + \frac{\kappa^2}{3}\rho_0 a_0^{-3}\mathrm{e}^{-3N} \ , \tag{47}$$

and $R = 3G'(N) + 12G(N) = 12H_0^2 + \kappa^2\rho_0 a_0^{-3}\mathrm{e}^{-3N}$, which can be solved with respect to N as follows,

$$N = -\frac{1}{3}\ln\left(\frac{\left(R - 12H_0^2\right)}{\kappa^2\rho_0 a_0^{-3}}\right) \ . \tag{48}$$

Eq.(45) takes the following form:

$$0 = 3\left(R - 9H_0^2\right)\left(R - 12H_0^2\right)\frac{d^2F(R)}{d^2R} - \left(\frac{1}{2}R - 9H_0^2\right)\frac{dF(R)}{dR} - \frac{1}{2}F(R)\,. \tag{49}$$

By changing the variable from R to x by $x = \frac{R}{3H_0^2} - 3$, Eq.(49) reduces to the hypergeometric differential equation:

$$0 = x(1-x)\frac{d^2F}{dx^2} + (\gamma - (\alpha + \beta + 1)\,x)\,\frac{dF}{dx} - \alpha\beta F\ . \tag{50}$$

Here

$$\gamma = -\frac{1}{2}\ , \alpha + \beta = -\frac{1}{6}\ , \quad \alpha\beta = -\frac{1}{6}\ , \tag{51}$$

Solution of (50) is given by Gauss' hypergeometric function $F(\alpha, \beta, \gamma; x)$:

$$F(x) = AF(\alpha, \beta, \gamma; x) + Bx^{1-\gamma}F(\alpha - \gamma + 1, \beta - \gamma + 1, 2 - \gamma; x)\ . \tag{52}$$

Here A and B are constant.

Thus, we demonstrated that modified $F(R)$ gravity may describe the ΛCDM epoch without the need to introduce the effective cosmological constant.

III. The formulation of modified gravity as General Relativity with generalized fluid and finite time future singularities

Bamba-Nojiri-SDO,JCAP 0810:045,2008; Nojiri-SDO, PRD 78,046006, 2008

Let us start from the general modified gravity with the action:

$$S = \int d^4x \sqrt{-g} \left\{ \frac{1}{2\kappa^2} \left(R + f\left(R, R_{\mu\nu}R^{\mu\nu}, R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}, \Box R, \Box^{-1}R, \cdots \right) \right) + L_{\mathrm{m}} \right\} , \tag{53}$$

where all combinations of local and non-local terms are possible, L_{m} is matter Lagrangian and the function $f(R, \cdots)$ may also contain gravitational partner (say, dilatons, axion, etc. in string-inspired gravity). In all cases for theory (53), it is possible to write the gravitational field equations in the form of standard FRW equations with effective energy-density ρ_{eff} and pressure p_{eff} produced by the extra gravitational terms $F(R, \cdots)$ and L_{m}.

For instance, when $f = f(R)$, one gets

$$\begin{aligned} \rho_{\rm eff} &= \frac{1}{\kappa^2}\left(-\frac{1}{2}f(R) + 3\left(H^2 + \dot{H}\right)f'(R)\right. \\ &\quad \left. -18\left(4H^2\dot{H} + H\ddot{H}\right)f''(R)\right) \\ &\quad +\rho_{\rm matter}\,, \qquad (54) \\ p_{\rm eff} &= \frac{1}{\kappa^2}\left(\frac{1}{2}f(R) - \left(3H^2 + \dot{H}\right)f'(R)\right. \\ &\quad +6\left(8H^2\dot{H} + 4\dot{H}^2 + 6H\ddot{H} + \dddot{H}\right)f''(R) \\ &\quad \left. +36\left(4H\dot{H} + \ddot{H}\right)^2 f'''(R)\right) \\ &\quad +p_{\rm matter}\,. \qquad (55) \end{aligned}$$

In case of Gauss-Bonnet modified gravity:

$$\rho_{\rm eff} = \frac{1}{2\kappa^2}\left[\mathcal{G} f_{\mathcal{G}}'(\mathcal{G}) - f_{\mathcal{G}}(\mathcal{G}) - 24^2 H^4 \left(2\dot{H}^2 + H\ddot{H} + 4H^2\dot{H}\right) f_{\mathcal{G}}''\right] + \rho_{\rm matter} ,$$

$$p_{\rm eff} = \frac{1}{2\kappa^2}\left[f_{\mathcal{G}}(\mathcal{G}) + 24^2 H^2 \left(3H^4 + 20H^2\dot{H}^2 + 6\dot{H}^3 + 4H^3\ddot{H} + H^2\dddot{H}\right) f_{\mathcal{G}}''(\mathcal{G}) - 24^3 H^5 \left(2\dot{H}^2 + H\ddot{H} + 4H^2\dot{H}\right)^2 f_{\mathcal{G}}'''(\mathcal{G})\right] + p_{\rm matter} . \quad (56)$$

In the same way one can get the effective gravitational pressure and energy density so that the equations of motion for arbitrary modified gravity can be rewritten in the universal FRW form typical for General Relativity:

$$\frac{3}{\kappa^2}H^2 = \rho_{\rm eff} , \quad p_{\rm eff} = -\frac{1}{\kappa^2}\left(2\dot{H} + 3H^2\right) . \quad (57)$$

There are just standard FRW gravitational equations.

FINITE-TIME FUTURE SINGULARITIES

- Type I ("Big Rip") : For $t \to t_s$, $a \to \infty$, $\rho_{\rm eff} \to \infty$ and $|p_{\rm eff}| \to \infty$. This also includes the case of $\rho_{\rm eff}$, $p_{\rm eff}$ being finite at t_s.
- Type II ("sudden") : For $t \to t_s$, $a \to a_s$, $\rho_{\rm eff} \to \rho_s$ and $|p_{\rm eff}| \to \infty$
- Type III : For $t \to t_s$, $a \to a_s$, $\rho_{\rm eff} \to \infty$ and $|p_{\rm eff}| \to \infty$
- Type IV : For $t \to t_s$, $a \to a_s$, $\rho_{\rm eff} \to 0$, $|p_{\rm eff}| \to 0$ and higher derivatives of H diverge.
  This also includes the case in which $p_{\rm eff}$ ($\rho_{\rm eff}$) or both of $p_{\rm eff}$ and $\rho_{\rm eff}$ tend to some finite values, while higher derivatives of H diverge.

Curing singularity with R^2-term: *M.Abdalla, Nojiri, SDO, CQG22,L35(2005)*

III A. Reconstructed $F(R)$ as an effective perfect fluid

Let us start with the modified gravity FRW equations written in the following form:

$$
\begin{aligned}
3H^2 &= \frac{1}{F'(R)}\left(\frac{1}{2}F(R) + 3H\partial_t F'(R)\right) - 3\dot{H}\,, \\
-3H^2 - 2\dot{H} &= -\frac{1}{F'(R)}\left(\frac{1}{2}F(R) + 2H\partial_t F'(R) + \partial_t^2 F'(R)\right) - \dot{H}\,,
\end{aligned} \tag{58}
$$

where

$$
\begin{aligned}
\rho &= \frac{1}{\kappa^2}\left[\frac{1}{F'(R)}\left(\frac{1}{2}F(R) + 3H\partial_t F'(R)\right) - 3\dot{H}\right] \\
p &= -\frac{1}{\kappa^2}\left[\frac{1}{F'(R)}\left(\frac{1}{2}F(R) + 2H\partial_t F'(R) + \partial_t^2 F'(R)\right) + \dot{H}\right]\,.
\end{aligned} \tag{59}
$$

Then, Eqs. (58) take the form of the usual FRW equations, where the EoS parameter for this dark fluid is defined by:

$$w = \frac{p}{\rho} = -\frac{\frac{1}{F'(R)}\left(\frac{1}{2}F(R) + 2H\partial_t F'(R) + \partial_t^2 F'(R)\right) + \dot{H}}{\frac{1}{F'(R)}\left(\frac{1}{2}F(R) + 3H\partial_t F'(R)\right) - 3\dot{H}} . \qquad (60)$$

The corresponding EoS may be written as follows:

$$p = -\rho - \frac{1}{\kappa^2}\left(4\dot{H} + \frac{1}{F'(R)}\partial_t^2 F'(R) - \frac{H}{F'(R)}\partial_t F'(R)\right) . \qquad (61)$$

The Ricci scalar is $R = 6(2H^2 + \dot{H})$, then $F(R)$ is a function of the Hubble parameter H and its derivative $\dot{H}$.
The form of the EoS is written as: $p = -\rho + g(H, \dot{H}, \ddot{H}, \cdots)$, where

$$\begin{aligned} g(H, \dot{H}, \ddot{H}, \cdots) &= -\frac{1}{\kappa^2}\Big(4\dot{H} + \partial_t^2(\ln F'(R)) + (\partial_t \ln F'(R))^2 \\ &\quad - H\partial_t \ln F'(R)\Big) . \end{aligned} \qquad (62)$$

Then, by combining the FRW equations, it yields the following differential equation:

$$\dot{H} + \frac{\kappa^2}{2} g(H, \dot{H}, \ddot{H}, \cdots) = 0 \,. \tag{63}$$

Hence, for a given cosmological model, the function g given in (62) may be seen as a function of cosmic time t, and then by the time-dependence of the Ricci scalar, the function g is rewritten in terms of R. Finally, the function $F(R)$ is recovered by the expression (62). In this sense, Eq. (63) combining with the expression (62) results in:

$$\frac{dx(t)}{dt} + x(t)^2 - H(t)x(t) = \dot{H}(t) \,, \tag{64}$$

where

$$x(t) = \frac{d\left(\ln F'(R(t))\right)}{dt} \,. \tag{65}$$

Let us consider the oscillating Universe where

$$H(t) = H_1 \cos \omega t \,. \tag{66}$$

Then, by introducing this expression in the equation (64), it yields,

$$x(t) = \frac{d(\ln F'(R(t))}{dt} = H_1 \cos \omega t \,. \tag{67}$$

Inverting the expression for the Ricci scalar $R = 6(2H^2 + \dot{H})$, and the expression (67), the corresponding $F(R)$ is obtained,

$$\begin{aligned} F(R) = & \\ & \left(48H_1^2 \omega \zeta^3(R)\right)^{-1} \left(24\omega^2(\omega - \sigma(R))\zeta(R) + (3\omega^2 - R + 24H_1^2)\sigma(R) \right. \\ & \left. +\sqrt{3}(4R - 3\omega^2) - 48\sqrt{3}H_1^2\omega\right) (\sqrt{3}\omega - \sigma(R))\mathrm{e}^{\zeta(R)} \,, \end{aligned} \tag{68}$$

$$\begin{aligned} \sigma(R) &= \sqrt{48H_1^2 + 3\omega^2 - 4R} \,, \\ \zeta(R) &= \frac{\sqrt{24H_1^2 + 3\omega^2 - \sqrt{3}\omega\sigma(R) - 2R}}{2\sqrt{6}\omega} \,. \end{aligned} \tag{69}$$

Hence, we have reconstructed the action for $F(R)$ that is able to reproduce a cyclic evolution (66).

Little Rip Universe: the Hubble parameter and the scale factor are,

$$H(t) = h_0 \mathrm{e}^{\alpha t} + h_1 \,, \quad \to \quad a(t) = a_0 \mathrm{e}^{4\beta \mathrm{e}^{\alpha t} + 6\alpha t} \,. \tag{70}$$

where $h_0 = 4\alpha\beta$ and $h_1 = 6\alpha$. It is straightforward to see that the function (70) describes a Universe, where for small times $t \ll \alpha$, the Hubble parameter can be approximated as a constant, reproducing a de Sitter solution, as in the case of ΛCDM model. For large times, the Universe ends in an eternal phantom phase, where the EoS parameter $w_{F(R)} < -1$. The $F(R)$ action that reproduces the solution (70) can be calculated is found to be

$$F(R) = \left[C_1 + C_2 \sqrt{4 \frac{R}{R_0} + 75} \right] \mathrm{e}^{\sqrt{\frac{R}{12 R_0} + \frac{25}{16}}} \,. \tag{71}$$

where $R_0 = \alpha^2$, $C_1 = -24\mathrm{e}^{-39/12} R_0$, and $C_2 = 2\sqrt{3} R_0$. Note that the action (71) turns out to be the Einstein-Hilbert action plus some corrections for small values of the Ricci curvature R,

$$F(R) \sim \kappa_1 R + \kappa_2 \frac{R^2}{R_0} + \kappa_3 \frac{R^3}{R_0^2} + \cdots \,. \tag{72}$$

Here the couplings κ_i are constants depending on $C_{1,2}$. Hence, for small values of the Ricci scalar the action reduces to the action for General Relativity plus power-law curvature corrections. Hence, the action (71) represents a viable model where GR can be recovered while the curvature scalar corrections remain small. It is important that such additional corrections become relevant close to the little rip evolution. In order to estimate the time for the little rip induced dissolution of bound structures in a naive way, one might compare the energy-density of a bound system as the Solar System with the density $\rho_{F(R)}$. For the model (70), such density can be approximated for large times by

$$\rho_{F(R)} = \rho_0 \mathrm{e}^{2\alpha t}\,, \tag{73}$$

where ρ_0 is a constant that can be set by imposing that the current value of the energy-density is $\rho_{F(R)}(t_0) = (3/\kappa^2)H_0^2 \sim 10^{-47}\,\mathrm{GeV}^4$, where the age of the Universe is taken to be $t_0 \sim 13.73\,\mathrm{Gyrs}$. One can set the time of the little rip dissolution occurrence when the gravitational coupling of the Sun-Earth system is broken due to the cosmological expansion. By assuming a mean density of the Sun-Earth system given by $\rho_{\odot-\oplus} = 0.594 \times 10^{-3}\,\mathrm{kg/m}^3 \sim 10^{-21}\,\mathrm{GeV}^4$, the time for the little rip dissolution of bound structures is,

$$t_{\mathrm{LR}} = 13.73\,\mathrm{Gyrs} + 29.93/\alpha\,. \tag{74}$$

Hence, depending on the parameter α, the appearance of the little rip may last shorter or longer. For example, when $\alpha = 10^{-1}\,\mathrm{Gyrs}^{-1}$, the little rip occurs at the Universe age of $t_{\mathrm{LR}} = 313.03\,\mathrm{Gyrs}$, while for $\alpha \geq 1\,\mathrm{Gyrs}^{-1}$, the time for the decoupling will be much shorter.

IV. Modified non-local-F(R) gravity as the key for the inflation and dark energy

Nojiri-SDO, PLB 569, 821, 2008

The starting action of the non-local gravity is given by

$$S = \int d^4x\sqrt{-g}\left\{\frac{1}{2\kappa^2}R\left(1 + f(\Box^{-1}R)\right) + \mathcal{L}_{\rm matter}\right\} . \qquad (75)$$

The above action can be rewritten by introducing two scalar fields ϕ and ξ:

$$\begin{aligned} S &= \int d^4x\sqrt{-g}\left[\frac{1}{2\kappa^2}\left\{R\left(1 + f(\phi)\right) + \xi\left(\Box\phi - R\right)\right\} + \mathcal{L}_{\rm matter}\right] \\ &= \int d^4x\sqrt{-g}\left[\frac{1}{2\kappa^2}\left\{R\left(1 + f(\phi)\right) - \partial_\mu\xi\partial^\mu\phi - \xi R\right\} + \mathcal{L}_{\rm matter}\right] \qquad (76) \end{aligned}$$

Varying (76) with respect to the metric tensor $g_{\mu\nu}$ gives

$$\begin{aligned} 0 &= \frac{1}{2}g_{\mu\nu}\left\{R\left(1+f(\phi)-\xi\right)-\partial_\rho\xi\partial^\rho\phi\right\}-R_{\mu\nu}\left(1+f(\phi)-\xi\right) \\ &\quad +\frac{1}{2}\left(\partial_\mu\xi\partial_\nu\phi+\partial_\mu\phi\partial_\nu\xi\right) \\ &\quad -\left(g_{\mu\nu}\Box-\nabla_\mu\nabla_\nu\right)\left(f(\phi)-\xi\right)+\kappa^2T_{\mu\nu}\ . \end{aligned} \tag{77}$$

On the other hand, the variation with respect to ϕ gives

$$0=\Box\xi+f'(\phi)R\ . \tag{78}$$

Now we assume the FRW metric

$$ds^2 = -dt^2 + a(t)^2 \sum_{i=1,2,3} \left(dx^i\right)^2 \ , \tag{79}$$

and the scalar fields ϕ and ξ only depend on time. Then Eq.(77) has the following form:

$$0 = -3H^2\left(1 + f(\phi) - \xi\right) + \frac{1}{2}\dot{\xi}\dot{\phi} - 3H\left(f'(\phi)\dot{\phi} - \dot{\xi}\right) + \kappa^2\rho \ , \tag{80}$$

$$\begin{aligned} 0 = &\left(2\dot{H} + 3H^2\right)\left(1 + f(\phi) - \xi\right) + \frac{1}{2}\dot{\xi}\dot{\phi} + \\ &\left(\frac{d^2}{dt^2} + 2H\frac{d}{dt}\right)\left(f(\phi) - \xi\right) + \kappa^2 p \ . \end{aligned} \tag{81}$$

On the other hand, scalar equations are:

$$0 = \ddot{\phi} + 3H\dot{\phi} + 6\dot{H} + 12H^2 \ , \tag{82}$$

$$0 = \ddot{\xi} + 3H\dot{\xi} - \left(6\dot{H} + 12H^2\right) f'(\phi) \ . \tag{83}$$

We now assume deSitter solution $H = H_0$, then Eq.(82) can be solved as

$$\phi = -4H_0 - \phi_0 \mathrm{e}^{-3H_0 t} + \phi_1 \ , \tag{84}$$

with constants of integration, ϕ_0 and ϕ_1. For simplicity, we only consider the case that $\phi_0 = \phi_1 = 0$. We also assume $f(\phi)$ is given by

$$f(\phi) = f_0 \mathrm{e}^{b\phi} = f_0 \mathrm{e}^{-4bH_0\phi} \ . \tag{85}$$

Then Eq.(83) can be solved as follows,

$$\xi = -\frac{3f_0}{3-4b}\mathrm{e}^{-4bH_0 t} + \frac{\xi_0}{3H_0}\mathrm{e}^{-3H_0 t} - \xi_1 \ . \tag{86}$$

Here ξ_0 and ξ_1 are constants. For the deSitter space a behaves as $a = a_0 \mathrm{e}^{H_0 t}$. Then for the matter with constant equation of state w, we find

$$\rho = \rho_0 \mathrm{e}^{-3(w+1)H_0 t} \ . \tag{87}$$

Then by substiruting (84), (86), and (87) into (80), we obtain

$$0 = -3H_0^2\left(1+\xi_1\right) + 6H_0^2 f_0\left(2b-1\right)\mathrm{e}^{-4H_0 bt} + \kappa^2\rho_0\mathrm{e}^{-3(w+1)H_0 t}\ . \quad (88)$$

When $\rho_0 = 0$, if we choose

$$b = \frac{1}{2}\ , \quad \xi_1 = -1\ , \quad (89)$$

deSitter space can be a solution. Even if $\rho \neq 0$, if we choose

$$b = \frac{3}{4}(1+w)\ , \quad f_0 = \frac{\kappa^2\rho_0}{3H_0^2\left(1+3w\right)}\ , \quad \xi_1 = -1\ , \quad (90)$$

there is a deSitter solution.

In the presence of matter with $w \neq 0$, we may have a deSitter solution $H = H_0$ even if $f(\phi)$ given by

$$f(\phi) = f_0\mathrm{e}^{\phi/2} + f_1\mathrm{e}^{3(w+1)\phi/4}\ . \quad (91)$$

Then the following solution exists:

$$\begin{aligned} \phi &= -4H_0 t \ , \\ \xi &= 1 + 3f_0 \mathrm{e}^{-2H_0 t} + \frac{f_1}{w} \mathrm{e}^{-3(w+1)H_0 t} \ , \\ \rho &= -\frac{3(3w+1)H_0^2 f_1}{\kappa^2} \mathrm{e}^{-3(1+w)H_0 t} \ . \end{aligned} \tag{92}$$

Note that H_0 in (84) can be arbitrary and can be determined by an initial condition. Since H_0 can be small or large, the theory with $b = 1/2$ could describe the early-time inflation or current cosmic acceleration.

Motivated by this, we may propose the following model:

$$f(\phi) = \begin{cases} f_0 \mathrm{e}^{\phi/2} & 0 > \phi > \phi_1 \\ f_0 \mathrm{e}^{\phi_1/2} & \phi_1 > \phi > \phi_2 \\ f_0 \mathrm{e}^{(\phi - \phi_2 + \phi_1)/2} & \phi < \phi_2 \end{cases} . \quad (93)$$

Here ϕ_1 and ϕ_2 are constants. We also assume that matter could be neglected when $0 > \phi > \phi_1$ or $\phi < \phi_2$. Since the above function $f(\phi)$ is not smooth around $\phi = \phi_1$ and ϕ_2, one may replace the above $f(\phi)$ with a more smooth function. When $0 > \phi > \phi_1$ or $\phi < \phi_2$, the universe is described by the deSitter solution although corresponding H_0 might be different.

When $\phi_1 > \phi > \phi_2$, since $f(\phi)$ is a constant, the universe is described by the Einstein gravity, where effective gravitational constant $\kappa_{\rm eff}$ is given by

$$\frac{1}{\kappa_{\rm eff}^2} = \frac{1}{\kappa^2}\left(1 + f_0 {\rm e}^{\phi_1/2}\right) . \tag{94}$$

Then due to the matter contribution there could occur matter dominated phase. In this phase, the Hubble rate H behaves as $H = \frac{2}{3(t_0+t)}$ with a constant t_0 and the scalar curvature is given by $R = \frac{4}{3(t_0+t)^2}$. Now we assume that the universe started at $t = 0$ with a rather big but constant curvature $R = R_I = 12H_I^2$ with a constant H_I, that is, the universe is in deSitter phase. Then in the model (93), by following (84), ϕ behaves as $\phi = -4H_I t$. Subsequently, at $t = t_1 \equiv -\phi_1/4H_I$, we have $\phi = \phi_1$ and the universe enters into the matter dominated phase. If the curvature is continuous at $t = t_1$, t_0 can be found by solving

$$R = \frac{4}{3\left(t_0 + t_1\right)^2} = 12H_I^2 . \tag{95}$$

If ϕ and $\dot{\phi}$ are also continuous, when $\phi_1 > \phi > \phi_2$, ϕ is given by solving (82) as

$$\phi = -\frac{4}{3}\ln\left(\frac{t}{t_1}\right) - \tilde{\phi}\,(t - t_1) + \phi_1\ , \quad \tilde{\phi} \equiv -4H_I\,(t_0 + t_1)^2 + \frac{4}{3}\,(t_0 + t_1)\ . \tag{96}$$

When $\phi = \phi_2$, the deSitter phase, which corresponds to the accelerating expansion of the present universe, could have started. The solution corresponds to deSitter space (with some shifts of parameters) and $H_0 = H_L$ could be given by solving

$$12H_L^2 = \frac{4}{3\,(t_0 + t_2)^2}\ . \tag{97}$$

if the curvature is continuous at $\phi = \phi_2$. In (97), t_2 is defined by $\phi(t_2) = \phi_2$. Thus, we got the cosmological FRW model with inflation, radiation/matter dominated phase, and current accelerating expansion.

V. Late-time cosmology in modified Gauss-Bonnet $f(G)$ gravity

Nojiri-SDO, Phys.Lett.B631,1,2006

Our example is modified Gauss-Bonnet gravity.

Let us start from the action :

$$S = \int d^4x\sqrt{-g}\left(\frac{1}{2\kappa^2}R + f(G) + \mathcal{L}_m\right) . \qquad (98)$$

Here $\mathcal{L}_m$ is the matter Lagrangian density and G is the GB invariant:
$G = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\xi\sigma}R^{\mu\nu\xi\sigma}$.

By variation over $g_{\mu\nu}$ one gets:

$$
\begin{aligned}
0 = & \frac{1}{2\kappa^2}\left(-R^{\mu\nu} + \frac{1}{2}g^{\mu\nu}R\right) + T^{\mu\nu} + \frac{1}{2}g^{\mu\nu}f(G) \\
& -2f'(G)RR^{\mu\nu} + 4f'(G)R^{\mu}{}_{\rho}R^{\nu\rho} - 2f'(G)R^{\mu\rho\sigma\tau}R^{\nu}{}_{\rho\sigma\tau} \\
& -4f'(G)R^{\mu\rho\sigma\nu}R_{\rho\sigma} + 2\left(\nabla^{\mu}\nabla^{\nu}f'(G)\right)R \\
& -2g^{\mu\nu}\left(\nabla^{2}f'(G)\right)R - 4\left(\nabla_{\rho}\nabla^{\mu}f'(G)\right)R^{\nu\rho} \\
& -4\left(\nabla_{\rho}\nabla^{\nu}f'(G)\right)R^{\mu\rho} + 4\left(\nabla^{2}f'(G)\right)R^{\mu\nu} \\
& +4g^{\mu\nu}\left(\nabla_{\rho}\nabla_{\sigma}f'(G)\right)R^{\rho\sigma} \\
& -4\left(\nabla_{\rho}\nabla_{\sigma}f'(G)\right)R^{\mu\rho\nu\sigma}. \qquad (99)
\end{aligned}
$$

The equation corresponding to the first FRW equation has the following form:

$$0 = -\frac{3}{\kappa^2} H^2 + G f'(G) - f(G) - 24 \dot{G} f''(G) H^3 + \rho_m \ , \qquad (100)$$

where ρ_m is the matter energy density. When $\rho_m = 0$, Eq. (100) has a deSitter universe solution where H, and therefore G, are constant. For $H = H_0$, with constant H_0, Eq. (100) turns into

$$0 = -\frac{3}{\kappa^2} H_0^2 + 24 H_0^4 f' \left(24 H_0^4 \right) - f \left(24 H_0^4 \right) \ . \qquad (101)$$

For a large number of choices of the function $f(G)$, Eq. (101) has a non-trivial ($H_0 \neq 0$) real solution for H_0 (deSitter universe).

We now consider the case $\rho_m \neq 0$. Assuming that the EoS parameter $w \equiv p_m/\rho_m$ for matter (p_m is the pressure of matter) is a constant then, by using the conservation of energy: $\dot{\rho}_m + 3H(\rho_m + p_m) = 0$, we find $\rho = \rho_0 a^{-3(1+w)}$. The function $f(G)$ is chosen as

$$f(G) = f_0 \left|G\right|^{\beta} \ , \tag{102}$$

with constant f_0 and β. If $\beta < 1/2$, $f(G)$ term becomes dominant compared with the Einstein term when the curvature is small. If we neglect the contribution from the Einstein term in (100), the following solution may be found

$$\begin{aligned} h_0 &= \frac{4\beta}{3(1+w)} \ , \\ a_0 &= \left[-\frac{f_0(\beta-1)}{(h_0-1)\,\rho_0}\left\{24\left|h_0^3\left(-1+h_0\right)\right|\right\}^{\beta}(h_0-1+4\beta)\right]^{-\frac{1}{3(1+w)}} \ . \end{aligned} \tag{103}$$

Then the effective EoS parameter $w_{\rm eff}$ is less than -1 if $\beta < 0$, and for $w > -1$ is

$$w_{\rm eff} = -1 + \frac{2}{3h_0} = -1 + \frac{1+w}{2\beta} \ , \qquad (104)$$

which is again less than -1 for $\beta < 0$. Thus, if $\beta < 0$, we obtain an effective phantom with negative h_0 even in the case when $w > -1$. Near this Big Rip , however, the curvature becomes dominant and then the Einstein term dominates, so that the $f(G)$ term can be neglected. Therefore, the universe behaves as $a = a_0 t^{2/3(w+1)}$ and as a consequence the Big Rip does not eventually occur. The phantom era is transient. Unification is againe possible.