Bosonizations of Supersymmetry $U_q(\widehat{sl}(N|1))$ and $U_{q,p}(\widehat{sl}(N|1))$ for an arbitrary level k

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Background in Physics

Supersymmetric t - J model

- 1- dimensional Quantum mechanics
- attempt to understand high-T_c superconductivity

$U_q(\widehat{sl}(2|1))$ for Level $k=1$

$$\Downarrow$$

$\boxed{U_q(\widehat{sl}(N|1)) \text{ for Level } k}$

Summary

- We construct a bosonization of superalgebra $U_q(\widehat{sl}(N|1))$ for an arbitrary level k.
- We propose a bosonization of vertex operator of $U_q(\widehat{sl}(N|1))$.
- We introduce elliptic algebra $U_{q,p}(\widehat{sl}(N|M))$ and construct a bosonization for $M=1$.

$\boxed{\text{KEY WORDS}}$

BOSONIZATION

SUPER ALGEBRA $\quad \widehat{sl}(\mathrm{MIN})$

ELLIPTIC ALGEBRA $\quad U_{qp}$

BOSONIZATION

Bosonization

- Realization by differential operator

  "x, $\frac{\partial}{\partial x}$" Boson

Bosonizations of $U_q(\mathfrak{g})$

Level k

Center $C \in U_q(\mathfrak{g})$

$$C = k \in \mathbb{C}$$

- Bosonizations for an arbitrary level k (Wakimoto)
  is completely different from
  those for level $k=1$ (Frenkel-Kac)

$\boxed{\text{Level } k = 1}$ Frenkel-Kac realization

[Frenkel, Kac, Invent. Math. 62, 23-66 (1980)]

q-Version [Frenkel, Jing, Proc. Natl. Acad. Sci. 85, 9373-9377 (1988)]

$\boxed{\text{Level } k : \text{general}}$ Wakimoto realization

[Wakimoto, CMP104, 605-609 (1986)]

q-Version
- [Shiraishi, Phys. Lett. A171, 243-248 (1992)]
- [Matsuo, CMP161, 33-48 (1994)]

- To understand that two realizations are completely different, we overview two realizations for $U_q(\widehat{sl}(2))$.

Quantum Group $U_q(\widehat{sl}(2))$

Drinfeld Realization

- $a_n, \ X_m^{\pm}, \ h, \ c \quad (n \in \mathbb{Z}_{\neq 0}, \ m \in \mathbb{Z})$
- $[a_m, a_n] = \dfrac{[2m]\,[cm]}{m} \delta_{m+n,0}$
- $(z_1 - q^{\pm 2} z_2) X^{\pm}(z_1) X^{\pm}(z_2)$

$$= (q^{\pm 2} z_1 - z_2) X^{\pm}(z_2) X^{\pm}(z_1)$$

$$\left([a] = \frac{q^a - q^{-a}}{q - q^{-1}}, \quad X^{\pm}(z) = \sum_{m \in \mathbb{Z}} X_m^{\pm} z^{-m-1} \right)$$

Quantum Group $U_q(\widehat{sl}(2))$

- $$[X^+(z_1), X^-(z_2)] = \frac{1}{(q-q^{-1})z_1 z_2} \times$$
$$\times \left(\delta\left(q^c \frac{z_2}{z_1}\right) \psi_+\left(q^{\frac{c}{2}} z_2\right) - \delta\left(q^{-c} \frac{z_2}{z_1}\right) \psi_-\left(q^{-\frac{c}{2}} z_2\right) \right)$$

- $$\psi^+\left(q^{\frac{c}{2}} z\right) = q^h \exp\left((q-q^{-1}) \sum_{m>0} a_m z^{-m} \right)$$

- $$\psi^-\left(q^{-\frac{c}{2}} z\right) = q^{-h} \exp\left(-(q-q^{-1}) \sum_{m>0} a_{-m} z^{m} \right)$$

$$\delta(z) = \sum_{m \in \mathbb{Z}} z^m$$

q-Frenkel-Kac realization $U_q(\widehat{sl(2)})$

Level $k = 1$ [Frenkel, Jing, Proc. Natl. Acad. Scie. 85 9373-9377 (1988)]

$$\cdot\ X^{\pm}(z) = \exp\left(\pm \sum_{m>0} \frac{a_{-m}}{[m]} q^{\mp\frac{m}{2}} z^{m}\right) \times \exp\left(\mp \sum_{m>0} \frac{a_m}{[m]} q^{\mp\frac{m}{2}} z^{-m}\right) e^{\alpha} z^{\pm\partial}$$

- a_m : Boson in Drinfeld realization

$$[a_m, a_n] = \frac{[2m]\ [m]}{m} \delta_{m+n,0}$$

q-Wakimoto realization

[Shiraishi, Phys. Lett. A171 243-248 (1992)]

$U_q(\widehat{sl}(2))$

Level k : general

Boson $\alpha_m, \beta_m, \gamma_m, Q_\beta, Q_\gamma$

$$[\alpha_m, \alpha_n] = \frac{[(k+2)m][2m]}{m} \delta_{m+n,0}$$

$$[\beta_m, \beta_n] = -\frac{[m]^2}{m} \delta_{m+n,0}$$

$$[\gamma_m, \gamma_n] = \frac{[m]^2}{m} \delta_{m+n,0}$$

$$[\beta_0, Q_\beta] = -1, \quad [\gamma_0, Q_\gamma] = 1.$$

q- Wakimoto realization

$U_q(\widehat{sl}(2))$

$$\beta(z) = -\sum_{m\neq 0} \frac{\beta_m}{[m]} z^{-m} + Q_\beta + \beta_0 \log z$$

$$\gamma(z) = -\sum_{m\neq 0} \frac{\gamma_m}{[m]} z^{-m} + Q_\gamma + \gamma_0 \log z$$

$$\beta_\pm(z) = \pm (q-q^{-1}) \sum_{\pm m>0} \beta_m z^{-m} \pm \beta_0 \log q$$

$$\alpha_\pm(z) = \pm (q-q^{-1}) \sum_{\pm m>0} \alpha_m z^{-m} \pm \alpha_0 \log q$$

q- Wakimoto realization

[Shiraishi, Phys. Lett. A171, 243-248 (1992)]

- $X^+(z) = \dfrac{1}{(q-q^{-1})z}\Big(:\exp\big(\beta_+(z) - (\beta+\gamma)(qz)\big): - :\exp\big(\beta_-(z) - (\beta+\gamma)(q^{-1}z)\big): \Big)$

- $X^-(z) = \dfrac{1}{(q-q^{-1})z}\Big(:\exp\big(\alpha_-(q^{-\frac{k+2}{2}}z) + \beta_-(q^{-k-2}z) + (\beta+\gamma)(q^{-k-1}z)\big): - :\exp\big(\alpha_+(q^{\frac{k+2}{2}}z) + \beta_+(q^{k+2}z) + (\beta+\gamma)(q^{k+1}z)\big): \Big)$

- $a_m = q^{-\frac{k+2}{2}|m|}\alpha_m + \left(q^{-k|m|} + q^{-(k+2)|m|}\right)\beta_m$

$U_q(\widehat{sl}(2))$

$\boxed{\text{q-Frenkel-Kac realization}}$

$$X^{\pm}(z) = :\exp(\alpha_{\pm}(z)):$$

$$\alpha_{\pm}(z) = \mp \sum_{m \neq 0} \frac{a_m}{[m]} q^{\mp \frac{m}{2}} z^{-m} + \alpha_{\pm}\partial$$

Level $k=1$

Bosonization of Level k is completely different from those of Level $k=1$.

SUPER ALGEBRA.

$\boxed{U_q(\widehat{sl}(2|1))}$ $e_{\bar{i}}, f_{\bar{i}}, h_{\bar{i}}$ $(\bar{i}=0,1,2)$

$$[h_{\bar{i}}, e_{\bar{j}}] = A_{\bar{i}\bar{j}} e_{\bar{j}}$$

$$[h_{\bar{i}}, f_{\bar{j}}] = -A_{\bar{i}\bar{j}} f_{\bar{j}}$$

$$[e_1, f_1] = \frac{q^{h_1} - q^{-h_1}}{q - q^{-1}}$$

$$\{e_{\bar{i}}, f_{\bar{i}}\} = \frac{q^{h_{\bar{i}}} - q^{-h_{\bar{i}}}}{q - q^{-1}} \quad (\bar{i}=0,2)$$

$$(A_{\bar{i}\bar{j}}) = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 0 \end{bmatrix}$$

$$[x, y] = xy - yx$$

$$\{x, y\} = xy + yx$$

[Yamane, Publ. RIMS 35, (1999)]

$\boxed{U_q(\widehat{sl}(2|1)}$

Serre relation

$$\{e_1, [e_1, e_2]_{q^{-1}}\}_q = 0$$

$$[e_0, [e_0, e_2]_{q^{-1}}]_q = 0$$

Extra Serre relation

$$\{e_2, [e_0, \{e_2 [e_0, e_1]_q\}]\}_{q^{-1}}$$

$$= \{e_0, [e_2, \{e_0, [e_2, e_1]_q\}]\}_{q^{-1}}$$

ELLIPTIC ALGEBRA

Quantum Group	Elliptic Algebra
$U_q(\mathfrak{g})$	$U_{q,p}(\mathfrak{g})$

$$U_q(\mathfrak{g}) \xrightarrow[\text{deformation}]{} U_{q,p}(\mathfrak{g})$$

$$(z_1 - q^2 z_2)\, X^+(z_1) X^+(z_2) = (q^2 z_1 - z_2)\, X^+(z_2) X^+(z_1)$$

$$[u_1 - u_2 + 1]_r F(z_1) F(z_2) = [u_1 - u_2 - 1]_r F(z_2) F(z_1)$$

Trigonometric function $(z = q^{2u})$ — Elliptic function

6-vertex model — ABF-face model

$\boxed{\text{KEY WORDS}}$ Now we have studied

Level k Bosonization

Superalgebra $U_q(\widehat{sl}(M|N))$

Elliptic algebra $U_{q,p}(\mathfrak{g})$

NEW RESULTS

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$\boxed{\text{Table of Free Field Realization}}$ $U_q(\mathfrak{g})$

(Bosonization)

$\boxed{\text{Level } k=1}$ q-Frenkel-Kac realization

$$\mathfrak{g} = (ADE)^{(1)},\ (BC)^{(1)},\ G_2^{(1)},\ \widehat{sl}(M|N),\ \widehat{osp}(2|2)^{(2)}$$

$\boxed{\text{Level } k : \text{general}}$ q-Wakimoto realization

$$\mathfrak{g} = \widehat{sl}(N),\ \widehat{sl}(N|1)$$

BOSONIZATIONS

$$\boxed{\text{Quantum Superalgebra } U_q(\widehat{sl}(M|N))}$$

[Kac, Adv. Math 26, 8-96 (1977)]

[J. Van. de. Leur, Commun. Alg. 17, 1815-1841 (1989)]

[Yamane, Publ. RIMS 35, 321-390 (1999)]

$$[x, y] = x \cdot y - y \cdot x$$

$$\{x, y\} = x \cdot y + y \cdot x$$

Superalgebra $\widehat{sl}(M|N)$

Cartan matrix $(A_{ij})_{0\le i,j\le M+N-1}$

$$
\left(\begin{array}{cccccccccccc}
0 & -1 & 0 & & & & & & & & 0 & 1 \\
-1 & 2 & -1 & & & & & & & & 0 & 0 \\
0 & -1 & 2 & & & & & & & & & \\
 & & & \ddots & & & & & & & & \\
 & & & & 2 & -1 & & & & & & \\
 & & & & -1 & 2 & -1 & & & & & \\
 & & & & & -1 & 0 & 1 & & & & \\
 & & & & & & 1 & -2 & 1 & & & \\
 & & & & & & & 1 & -2 & & & \\
 & & & & & & & & & \ddots & & \\
0 & 0 & & & & & & & & -2 & 1 & 0 \\
0 & 0 & & & & & & & & 1 & -2 & 1 \\
1 & 0 & & & & & & & & 0 & 1 & -2
\end{array}\right)
$$

Quantum Superalgebra $U_q(\widehat{sl}(M|N))$

Drinfeld's generators

$$X_{i,m}^{\pm}, \quad a_{i,n}, \quad h_i, \quad c \qquad (1 \le i \le M+N-1,\ m \in \mathbb{Z},\ n \in \mathbb{Z}_{\neq 0})$$

Generating functions

$$X_i^{\pm}(z) = \sum_{m \in \mathbb{Z}} X_{i,m}^{\pm} z^{-m-1} \qquad (1 \le i \le M+N-1)$$

$$\psi_i^{+}(q^{\frac{c}{2}} z) = q^{h_i} \exp\left((q-q^{-1}) \sum_{m>0} a_{i,m} z^{-m} \right)$$

$$\psi_i^{-}(q^{-\frac{c}{2}} z) = q^{-h_i} \exp\left(-(q-q^{-1}) \sum_{m>0} a_{i,-m} z^{m} \right)$$

Quantum Superalgebra $U_q(\widehat{sl}(M|N))$

Defining relations

- $[a_{m,i}, a_{n,j}] = \dfrac{[km][A_{ij}m]}{m} q^{-k|m|} \delta_{m+n,0}$,
- $[a_{m,i}, X_j^+(z)] = \dfrac{[A_{ij}m]}{m} q^{-k|m|} z^m X_j^+(z)$,
- $[a_{m,i}, X_j^-(z)] = -\dfrac{[A_{ij}m]}{m} z^m X_j^-(z)$.

Quantum Superalgebra $U_q(\widehat{sl}(M|N))$

$\{x, y\} = xy + yx$

Defining relations

for $|A_{ij}| \neq 0$

- $(z_1 - q^{\pm A_{ij}} z_2)\, X_i^{\pm}(z_1) X_j^{\pm}(z_2) = (q^{\pm A_{ji}} z_1 - z_2)\, X_j^{\pm}(z_2) X_i^{\pm}(z_1)$
- $[\, X_i^{\pm}(z_1),\, X_j^{\pm}(z_2)\,] = 0$ for $|A_{ij}| = 0$, $(i, j) \neq (M, M)$
- $\{ X_M^{\pm}(z_1),\, X_M^{\pm}(z_2) \} = 0$
- $[\, X_i^{+}(z_1),\, X_j^{-}(z_2)\,] = \dfrac{\delta_{ij}}{(q - q^{-1}) z_1 z_2} \times$ for $(i, j) \neq (M, M)$

  $\times \left(\delta\!\left(q^{c} \frac{z_2}{z_1}\right) \psi_i^{+}(q^{\frac{c}{2}} z_2) - \delta\!\left(q^{-c} \frac{z_2}{z_1}\right) \psi_i^{-}(q^{-\frac{c}{2}} z_2) \right)$
- $\{ X_M^{+}(z_1),\, X_M^{-}(z_2) \} = \dfrac{1}{(q - q^{-1}) z_1 z_2} \times$

  $\times \left(\delta\!\left(q^{c} \frac{z_2}{z_1}\right) \psi_M^{+}(q^{\frac{c}{2}} z_2) - \delta\!\left(q^{-c} \frac{z_2}{z_1}\right) \psi_M^{-}(q^{-\frac{c}{2}} z_2) \right)$

Quantum Superalgebra $U_q(\widehat{sl}(M|N))$

Serre relations

- $\left(X_i^{\pm}(z_1) X_i^{\pm}(z_2) X_j^{\pm}(z) - (q+q^{-1}) X_i^{\pm}(z_1) X_j^{\pm}(z) X_i^{\pm}(z_2) + X_j^{\pm}(z) X_i^{\pm}(z_1) X_i^{\pm}(z_2) \right) + (z_1 \leftrightarrow z_2) = 0 \quad$ for $|A_{ij}| = 1,\ i \neq M$

- $$\begin{aligned}
&\Big(X_M^{\pm}(z_1) X_{M+1}^{\pm}(w_1) X_M^{\pm}(z_2) X_{M-1}^{\pm}(w_2) - q^{-1} X_M^{\pm}(z_1) X_{M+1}^{\pm}(w_1) X_{M-1}^{\pm}(w_2) X_M^{\pm}(z_2) \\
&- q X_M^{\pm}(z_1) X_M^{\pm}(z_2) X_M^{\pm}(w_2) X_{M+1}^{\pm}(w_1) + X_M^{\pm}(z_1) X_{M-1}^{\pm}(w_2) X_M^{\pm}(z_2) X_{M+1}^{\pm}(w_1) \\
&+ X_{M+1}^{\pm}(w_1) X_M^{\pm}(z_2) X_{M-1}^{\pm}(w_2) X_M^{\pm}(z_1) - q^{-1} X_{M+1}^{\pm}(w_1) X_{M-1}^{\pm}(w_2) X_M^{\pm}(z_2) X_M^{\pm}(z_1) \\
&- q X_M^{\pm}(z_2) X_{M-1}^{\pm}(w_2) X_{M+1}^{\pm}(w_1) X_M^{\pm}(z_1) + X_{M-1}^{\pm}(w_2) X_M^{\pm}(z_2) X_{M+1}^{\pm}(w_1) X_M^{\pm}(z_1) \Big) \\
&+ (z_1 \leftrightarrow z_2) = 0
\end{aligned}$$

Free Field Realization

$U_q(\widehat{sl}(N|1))$

Boson $a_m^i \ (1 \le i \le N)$ $\quad b_m^{ij}, \ c_m^{ij} \ (1 \le i < j \le N+1)$

$$\cdot\ [a_m^i, a_n^j] = \frac{[(k+N-1)m]\,[A_{ij}m]}{m}\,\delta_{m+n,0}$$

$$\cdot\ [b_m^{ij}, b_n^{i'j'}] = -\nu_i \nu_j \frac{[m]^2}{m}\,\delta_{ii'}\,\delta_{jj'}\,\delta_{m+n,0}$$

$$\cdot\ [c_m^{ij}, c_n^{i'j'}] = \nu_i \nu_j \frac{[m]^2}{m}\,\delta_{ii'}\,\delta_{jj'}\,\delta_{m+n,0}$$

$\nu_1 = \nu_2 = \cdots = \nu_N = +, \quad \nu_{N+1} = -$

[Kojima, JMP 53, 013515 (15 pages) (2012)] $\widehat{sl}(N|1)$

[Awata, Odake, Shiraishi, LMP 42, 271-279 (1997)] $\widehat{sl}(2|1)$

$\boxed{\text{Free Field Realization}}$

$\boxed{\text{Zero-Mode}}$ $\quad Q_b^{\bar{i}\bar{j}}, \; Q_c^{\bar{i}\bar{j}}$

$$\cdot \left[b_0^{\bar{i}\bar{j}}, Q_b^{\bar{i}'\bar{j}'} \right] = - \nu_{\bar{i}} \nu_{\bar{j}} \delta_{\bar{i}\bar{i}'} \delta_{\bar{j}\bar{j}'}$$

$$\cdot \left[c_0^{\bar{i}\bar{j}}, Q_c^{\bar{i}'\bar{j}'} \right] = \nu_{\bar{i}} \nu_{\bar{j}} \delta_{\bar{i}\bar{i}'} \delta_{\bar{j}\bar{j}'}$$

Cocycle Condition

$$\cdot \left[Q_b^{\bar{i}\bar{j}}, Q_b^{\bar{i}'\bar{j}'} \right] = \delta_{\bar{j}\, N+1} \delta_{\bar{j}'\, N+1} \log\left(\pi\sqrt{-1}\right)$$

Free Field Realization

- $b^{\bar{i}\bar{j}}(z) = -\sum_{m\neq 0} \frac{b_m^{\bar{i}\bar{j}}}{[m]} z^{-m} + Q_b^{\bar{i}\bar{j}} + b_0^{\bar{i}\bar{j}} \log z$
- $c^{\bar{i}\bar{j}}(z) = -\sum_{m\neq 0} \frac{c_m^{\bar{i}\bar{j}}}{[m]} z^{-m} + Q_c^{\bar{i}\bar{j}} + c_0^{\bar{i}\bar{j}} \log z$
- $b_\pm^{\bar{i}\bar{j}}(z) = \pm(q-q^{-1}) \sum_{\pm m>0} b_m^{\bar{i}\bar{j}} z^{-m} \pm b_0^{\bar{i}\bar{j}} \log q$
- $a_\pm^{\bar{i}}(z) = \pm(q-q^{-1}) \sum_{\pm m>0} a_m^{\bar{i}} z^{-m} \pm a_0^{\bar{i}} \log q$

Free Field Realization

$(1 \le i \le N-1)$

- $$X_i^+(z) = \frac{1}{(q-q^{-1})z} \sum_{j=1}^{i} \Big(:\exp\Big((b+c)^{ji}(q^{j-1}z) + b_+^{j\,i+1}(q^{j-1}z) - (b+c)^{j\,i+1}(q^{j}z) + \sum_{l=1}^{j-1} \big(b_+^{l\,i+1}(q^{l-1}z) - b_+^{li}(q^{l}z) \big) \Big): - :\exp\Big((b+c)^{ji}(q^{j-1}z) + b_-^{j\,i+1}(q^{j-1}z) - (b+c)^{j\,i+1}(q^{j-2}z) + \sum_{l=1}^{j-1} \big(b_+^{l\,i+1}(q^{l-1}z) - b_+^{li}(q^{l}z) \big) \Big): \Big).$$

- $$X_N^+(z) = \sum_{j=1}^{N} :\exp\Big((b+c)^{jN}(q^{j-1}z) + b^{j\,N+1}(q^{j-1}z) - \sum_{l=1}^{j-1} \big(b_+^{l,N+1}(q^{l}z) + b_+^{lN}(q^{l}z) \big) \Big): .$$

$$U_q(\widehat{sl}(N+1)) \longrightarrow U_q(\widehat{sl}(N|1))$$

Free Field Realization

$(1 \le i \le N-1)$

$$
\begin{aligned}
\cdot\ X_i^-(z) = \frac{1}{(q-q^{-1})z} \Big\{ \sum_{j=1}^{i-1} :\exp\Big(& a_-^{i}(q^{-\frac{k+N-1}{2}}z) + (b+c)^{j\,i+1}(q^{-k-j}z) \\
& + \sum_{l=j+1}^{i} \big(b_-^{l\,i+1}(q^{-k-l+1}z) - b_-^{l\,i}(q^{-k-l}z)\big) + \sum_{l=i+1}^{N} \big(b_-^{i\,l}(q^{-k-l}z) - b_-^{i+1\,l}(q^{-k-l-1}z)\big) + b_-^{i\,N+1}(q^{-k-N}z) - b_-^{i+1\,N+1}(q^{-k-N+1}z) \Big) \\
\times \Big(& \exp\big(-b_-^{j\,i}(q^{-k-j}z) - (b+c)^{j\,i}(q^{-k-j+1}z)\big) - \exp\big(-b_+^{j\,i}(q^{-k-j}z) - (b+c)^{j\,i}(q^{-k-j-1}z)\big) \Big): \\
- \sum_{j=i+1}^{N-1} :\exp\Big(& a_+^{i}(q^{\frac{k+N-1}{2}}z) + (b+c)^{i\,j+1}(q^{k+j}z) \\
& + \sum_{l=j+1}^{N} \big(b_+^{i\,l}(q^{k+l}z) - b_+^{i+1\,l}(q^{k+l-1}z)\big) + b_+^{i\,N+1}(q^{k+N}z) - b_+^{i+1\,N+1}(q^{k+N-1}z) \Big) \\
\times \Big(& \exp\big(b_+^{i+1\,j+1}(q^{k+j}z) - (b+c)^{i+1\,j+1}(q^{k+j+1}z)\big) - \exp\big(b_-^{i+1\,j+1}(q^{k+j}z) - (b+c)^{i+1\,j+1}(q^{k+j-1}z)\big) \Big): +
\end{aligned}
$$

$\cdots$

Free Field Realization

...

$$
\begin{aligned}
&+ : \exp\Big(a_-^{\bar{i}}(q^{-\frac{k+N-1}{2}}z) + (b+c)^{\bar{i}\,\bar{i}+1}(q^{-k-\bar{i}}z) + \sum_{l=\bar{i}+1}^{N} \big(b_-^{\bar{i}\,l}(q^{-k-l}z) - b_-^{\bar{i}+1\,l}(q^{-k-l+1}z) \big) \\
&\qquad + b_-^{\bar{i}\,N+1}(q^{-k-N}z) - b_-^{\bar{i}+1\,N+1}(q^{-k-N+1}z) \Big) : \\
&- : \exp\Big(a_+^{\bar{i}}(q^{\frac{k+N-1}{2}}z) + (b+c)^{\bar{i}\,\bar{i}+1}(q^{k+\bar{i}}z) + \sum_{l=\bar{i}+1}^{N} \big(b_+^{\bar{i}\,l}(q^{k+l}z) - b_+^{\bar{i}+1\,l}(q^{k+l-1}z) \big) \\
&\qquad + b_+^{\bar{i}\,N+1}(q^{k+N}z) - b_+^{\bar{i}+1\,N+1}(q^{k+N-1}z) \Big) : \Big\} \\
&+ : \exp\Big(a_+^{\bar{i}}(q^{\frac{k+N-1}{2}}z) - b^{\bar{i}\,N+1}(q^{k+N-1}z) - b_+^{\bar{i}+1\,N+1}(q^{k+N-1}z) + b_+^{\bar{i}+1\,N+1}(q^{k+N}z) \Big) : .
\end{aligned}
$$

Free Field Realization

$$\bullet\ X_N^-(z) = \frac{1}{(q-q^{-1})z}\Big\{ \sum_{j=1}^{N-1}\Big(-q^{j-1} : \exp\Big(a_-^N(q^{-\frac{k+N-1}{2}}z)$$

$$-b_-^{jN}(q^{-k-j}z) - (b+c)^{jN}(q^{-k-j+1}z) - b_-^{j\,N+1}(q^{-k-j}z) - b^{j\,N+1}(q^{-k-j+1}z) - \sum_{l=j+1}^{N-1}\big(b_-^{lN}(q^{-k-l}z) + b_-^{l\,N+1}(q^{-k-l}z)\big)\Big):$$

$$+q^{j-1} : \exp\Big(a_-^N(q^{-\frac{k+N-1}{2}}z)$$

$$-b_+^{jN}(q^{-k-j}z) - (b+c)^{jN}(q^{-k-j-1}z) - b_+^{j\,N+1}(q^{-k-j}z) - b^{j\,N+1}(q^{-k-j-1}z) - \sum_{l=j+1}^{N-1}\big(b_-^{lN}(q^{-k-l}z) + b_-^{l\,N+1}(q^{-k-l}z)\big)\Big):$$

$$-q^{N-1} : \exp\Big(a_-^N(q^{-\frac{k+N-1}{2}}z) - b^{N,N+1}(q^{-k-N+1}z)\Big):$$

$$+q^{N-1} : \exp\Big(a_+^N(q^{\frac{k+N-1}{2}}z) - b^{NN+1}(q^{k+N-1}z)\Big): \Big\}.$$

Free Field Realization

- $$a_{\bar{i}m} = q^{-\frac{k+N-1}{2}|m|} a_m^{\bar{i}} + \sum_{\ell=1}^{\bar{i}} \left(q^{-(k+\ell-1)|m|} b_m^{\ell \bar{i}+1} - q^{-(k+\ell)|m|} b_m^{\ell \bar{i}} \right) + \sum_{\ell=\bar{i}+1}^{N} \left(q^{-(k+\ell)|m|} b_m^{\bar{i}\ell} - q^{-(k+\ell-1)|m|} b_m^{\bar{i}+1,\ell} \right) + q^{-(k+N)|m|} b_m^{\bar{i} N+1} - q^{-(k+N-1)|m|} b_m^{\ell N+1},$$

- $$a_{N,m} = q^{-\frac{k+N-1}{2}|m|} a_m^{N} - \sum_{\ell=1}^{N-1} \left(q^{-(k+\ell)|m|} b_m^{\ell,N} + q^{-(k+\ell)|m|} b_m^{\ell,N+1} \right).$$

We have constructed the bosonization of supersymmetry $U_q(\widehat{sl}(N|1))$ for level k, by using three kind of bosons

$$a_m^i, \quad b_m^{ij}, \quad c_m^{ij}.$$

SUPPORTING ARGUMENT

$U_q(sl(N|1))$ Heisenberg Realization

- $$x_{ij} = \begin{cases} z_{ij} & (1 \le i < j \le N) \\ \theta_{i,N+1} & (1 \le i \le N,\ j = N+1) \end{cases}$$

$$z_{ij} z_{kl} = z_{kl} z_{ij}$$
$$\theta_{i,N+1}\,\theta_{j,N+1} = -\,\theta_{j,N+1}\,\theta_{i,N+1}$$

- $$\vartheta_{ij} = x_{ij} \frac{\partial}{\partial x_{ij}}$$

[Awata, Odake, Shiraishi, LMP42, 271-279 (1997)]
[Kimura, Preprint (1996)]

$U_q(sl(N|1))$ Heisenberg Realization

$$H_{\bar{i}} = \sum_{\bar{j}=1}^{N} H_{\bar{i}\bar{j}}, \quad E_{\bar{i}} = \sum_{\bar{j}=1}^{\bar{i}} E_{\bar{i}\bar{j}}, \quad F_{\bar{i}} = \sum_{\bar{j}=1}^{N} F_{\bar{i}\bar{j}}. \qquad (\lambda_{\bar{i}} \in \mathbb{C})$$

$$H_{\bar{i}\bar{j}} = \begin{cases} \nu_{\bar{i}} \vartheta_{\bar{j},\bar{i}} - \nu_{\bar{i}+1} \vartheta_{\bar{j},\bar{i}+1} & (1 \le \bar{j} \le \bar{i}-1) \\ \lambda_{\bar{i}} - (\nu_{\bar{i}} + \nu_{\bar{i}+1}) \vartheta_{\bar{i},\bar{i}+1} & (\bar{j} = \bar{i}) \\ \nu_{\bar{i}+1} \vartheta_{\bar{i}+1,\bar{j}+1} - \nu_{\bar{i}} \vartheta_{\bar{i},\bar{j}+1} & (\bar{i}+1 \le \bar{j} \le N) \end{cases}$$

$$E_{\bar{i}\bar{j}} = \frac{x_{\bar{j}\bar{i}}}{x_{\bar{j}\,\bar{i}+1}} [\vartheta_{\bar{j},\bar{i}+1}] \, q^{\sum_{l=1}^{\bar{j}-1} (\nu_{\bar{i}} \vartheta_{l\bar{i}} - \nu_{\bar{i}+1} \vartheta_{l,\bar{i}+1})}$$

$$
F_{\bar{i}\bar{j}} = \begin{cases}
v_{\bar{i}} \dfrac{x_{\bar{j}\,\bar{i}+1}}{x_{\bar{j}\bar{i}}} [\vartheta_{\bar{j}\bar{i}}]\, q^{\sum_{\ell=\bar{j}+1}^{\bar{i}-1} (V_{\bar{i}+1} \vartheta_{\ell,\bar{i}+1} - V_{\bar{i}} \vartheta_{\ell\bar{i}})} \times \\
\quad \times\, q^{-\lambda_{\bar{i}} + (V_{\bar{i}} + V_{\bar{i}+1}) \vartheta_{\bar{i}\,\bar{i}+1} + \sum_{\ell=\bar{i}+2}^{N+1} (V_{\bar{i}} \vartheta_{\bar{i}\ell} - V_{\bar{i}+1} \vartheta_{\bar{i}+1\,\ell})} & (1 \le \bar{j} \le \bar{i}-1) \\
x_{\bar{i},\bar{i}+1} \left[\lambda_{\bar{i}} - V_{\bar{i}} \vartheta_{\bar{i}\,\bar{i}+1} - \sum_{\ell=\bar{i}+2}^{N+1} (V_{\bar{i}} \vartheta_{\bar{i},\ell} - V_{\bar{i}+1} \vartheta_{\bar{i}+1,\ell}) \right] & (\bar{j} = \bar{i}) \\
-v_{\bar{i}+1} \dfrac{x_{\bar{i}\,\bar{j}+1}}{x_{\bar{i}+1\,\bar{j}+1}} [\vartheta_{\bar{i}+1,\bar{j}+1}]\, q^{\lambda_{\bar{i}} + \sum_{\ell=\bar{j}+1}^{N+1} (V_{\bar{i}+1} \vartheta_{\bar{i}+1,\ell} - V_{\bar{i}} \vartheta_{\bar{i}\ell})} & (\bar{i}+1 \le \bar{j} \le N)
\end{cases}
$$

$\boxed{\text{Replacement}}$ $\qquad (z \mapsto q^{\star} z)$

$$g_{ij} \longmapsto -b^{ij}_{\pm}(z)/\log q \qquad (1 \leq i < j \leq N+1)$$

$$[g_{ij}] \longmapsto \begin{cases} \dfrac{e^{\pm b^{ij}_{+}(z)} - e^{\pm b^{ij}_{-}(z)}}{(q-q^{-1})z} & (j \neq N+1) \\ 1 & (j = N+1) \end{cases}$$

$$x_{ij} \longmapsto \begin{cases} :e^{(b+c)^{ij}(z)}: & (j \neq N+1) \\ :e^{-b^{ij}(z)}: \quad \text{or} \quad :e^{-b^{ij}_{\pm}(z) - b^{ij}(z)}: & (j = N+1) \end{cases}$$

$$\lambda_i \longmapsto a^{i}_{\pm}(z)/\log q \qquad (1 \leq i \leq N)$$

$$[\lambda_i] \longmapsto \frac{e^{\pm a^{i}_{+}(z)} - e^{\pm a^{i}_{-}(z)}}{(q-q^{-1})z} \qquad (1 \leq i \leq N)$$

Example

$$F_{N,\bar{j}} \longmapsto \frac{1}{(q-q^{-1})z} \times$$

$$\times : \exp\Big(-a_-^{N}(z) - b^{\bar{j},N+1}(z) - (b+c)^{\bar{j}N}(z) + \sum_{l=\bar{j}+1}^{N-1}\big(b_-^{l,N+1}(z) - b_-^{l,N}(z)\big)\Big)$$

$$\times\Big(\exp\big(-b_+^{\bar{j}N}(z) - b^{\bar{j},N+1}(z)\big) - \exp\big(-b_-^{\bar{j},N}(z) - b^{\bar{j},N+1}(z)\big)\Big) : .$$

$$\longmapsto \frac{1}{(q-q^{-1})z} \times$$

$$\times : \exp\Big(-a_-^{N}\big(q^{-\frac{k+N-1}{2}}z\big) - b^{\bar{j},N+1}\big(q^{-k-\bar{j}+1}z\big) - (b+c)^{\bar{j}N}\big(q^{-k-\bar{j}+1}z\big) + \sum_{l=\bar{j}+1}^{N-1}\big(b_-^{lN+1}(q^{-k-l}z) - b_-^{lN}(q^{-k-l}z)\big)\Big)$$

$$\times\Big(\exp\big(-b_+^{\bar{j}N}(q^{-k-\bar{j}}z) - b^{\bar{j}N+1}(q^{-k-\bar{j}-1}z)\big) - \exp\big(-b_-^{\bar{j}N}(q^{-k-\bar{j}}z) - b^{\bar{j}N+1}(q^{-k-\bar{j}+1}z)\big)\Big) : .$$

$$= \frac{1}{(q-q^{-1})z}\Big(X^-_{N,2\bar{j}}(z) - X^-_{N,2\bar{j}-1}(z)\Big).$$

WAKIMOTO REALIZATION

Highest Weight Representation $V(\lambda)$

$$\boxed{V(\lambda) \subset F(p_a) = \bigoplus_{\substack{p_b^{ij} = -p_c^{ij} \in \mathbb{Z} \ (1 \leq i < j \leq N) \\ p_b^{i\,N+1} \in \mathbb{Z} \ (1 \leq i \leq N)}} F(p_a, p_b, p_c)}$$

- $|\lambda\rangle = |p_a, 0, 0\rangle$: highest weight vector with $\bar{\lambda} = \sum_{j=1}^{N} p_a^j \bar{\Lambda}_j$

- $|p_a, p_b, p_c\rangle$

$$= \exp\left(\sum_{i,j=1}^{N} \frac{\mathrm{Min}(i,j)(N-1-\mathrm{Max}(i,j))}{(N-1)(-k+N-1)} p_a^i Q_a^j - \sum_{1 \leq i < j \leq N+1} p_b^{ij} Q_b^{ij} + \sum_{1 \leq i < j \leq N} p_c^{ij} Q_c^{ij} \right) |0\rangle.$$

- $F(p_a, p_b, p_c)$ is generated by bosons $a_m^i, b_m^{ij}, c_m^{ij}$ on $|p_a, p_b, p_c\rangle$.

$\boxed{\text{Ghoost Boson Problem}}$

- Wakimoto realization $\neq$ Verma module
- Character of Wakimoto realization $=$ Character of Verma module

It is expected that $F(p_a)$ is Wakimoto realization. But ....

- Character of $F(p_a)$ $\neq$ Character of Verma module
- We would like to construct Wakimoto realization from $F(p_a)$.

$\boxed{\text{Wakimoto Realization}}$

- $F(p_a) = \bigoplus_{\substack{p_b^{ij} = -p_c^{ij} \in \mathbb{Z} \ (1 \le i < j \le N) \\ p_b^{i\,N+1} \in \mathbb{Z} \ (1 \le i \le N)}} F(p_a, p_b, p_c)$ is very large.

- There exists non-trivial (non-constant) operator $\eta_0 \xi_0$ that commutes with U_q.

$$[\eta_0 \xi_0 , U_q(\widehat{sl}(N|1))] = 0$$

ξ-η system

$$[\eta_0\xi_0, U_q] = 0$$

$$\left(\begin{aligned} \eta^{\bar{i}\bar{j}}(z) &= \sum_{m\in\mathbb{Z}} \eta^{\bar{i}\bar{j}}_m z^{-m-1} =: \exp(c^{\bar{i}\bar{j}}(z)): \\ \xi^{\bar{i}\bar{j}}(z) &= \sum_{m\in\mathbb{Z}} \xi^{\bar{i}\bar{j}}_m z^{-m} =: \exp(-c^{\bar{i}\bar{j}}(z)): \end{aligned} \right. \quad (1\le \bar{i} < \bar{j} \le N)$$

$$\left(\begin{aligned} &\{\eta^{\bar{i}\bar{j}}_m, \xi^{\bar{i}\bar{j}}_n\} = \delta_{m+n,0}, \quad \{\eta^{\bar{i}\bar{j}}_m, \eta^{\bar{i}\bar{j}}_n\} = \{\xi^{\bar{i}\bar{j}}_m, \xi^{\bar{i}\bar{j}}_n\} = 0 \\ &[\eta^{\bar{i}\bar{j}}_m, \xi^{\bar{k}\bar{l}}_n] = [\eta^{\bar{i}\bar{j}}_m, \eta^{\bar{k}\bar{l}}_n] = [\xi^{\bar{i}\bar{j}}_m, \xi^{\bar{k}\bar{l}}_n] = 0 \quad (\bar{i},\bar{j}) \neq (\bar{k},\bar{l}) \end{aligned} \right.$$

$$\left(\begin{aligned} \mathrm{Im}(\eta^{\bar{i}\bar{j}}_0) &= \mathrm{Ker}(\eta^{\bar{i}\bar{j}}_0) \\ \mathrm{Im}(\xi^{\bar{i}\bar{j}}_0) &= \mathrm{Ker}(\xi^{\bar{i}\bar{j}}_0) \end{aligned} \right.$$

Wakimoto Realization

$\widetilde{\mathcal{F}}(p_a)$

$$[\eta_0\xi_0, U_q] = 0, \quad \xi_0 = \prod_{1\le i<j\le N} \xi_0^{ij}, \quad \eta_0 = \prod_{1\le i<j\le N} \eta_0^{ij}$$

- $\widetilde{\mathcal{F}}(p_a) = \eta_0\xi_0 F(p_a)$ $\qquad U_q(\widehat{sl}(N|1))$ module

$$= \bigcap_{1\le i<j\le N} \mathrm{Ker}(\eta_0^{ij})$$

- Character of $\widetilde{\mathcal{F}}(p_a)$ coincides with those of Verma module.
- We call $\widetilde{\mathcal{F}}(p_a)$ Wakimoto Realization.

SCREENING

$\boxed{\text{Screening}}$ $Q_j \quad (1 \leq j \leq N)$

- Our realization is very large.
- There exists nontrivial (non-constant) operator that commutes with $U_q(\widehat{sl}(N|1))$.

$$[Q_j, U_q(\widehat{sl}(N|1))] = 0$$

$$[Q_j, \eta_0 \xi_0] = 0$$

- For level $k = 1$ there doesn't exist such a operator.

$\boxed{\text{Screening}}$ $\quad Q_j \ (1 \leq j \leq N) \qquad k \neq -N+1$

$$\boxed{Q_j = \int_0^{s\infty} S_j(z)\, d_p z \qquad (p = q^{2(k+N-1)})}$$

Jackson integral

$$\int_0^{s\infty} f(z)\, d_p z = s(1-p) \sum_{m \in \mathbb{Z}} f(sp^m)\, p^m .$$

$$\boxed{\begin{aligned} &[\,Q_j,\ U_q(\widehat{sl}(N|1))\,] = 0 \\ &[\,Q_j,\ \eta_0 \xi_0\,] = 0 \end{aligned}}$$

Screening

$$k \neq -N+1$$

$$\boxed{S_{\bar{j}}(z) =: \exp\left(-\left(\frac{1}{k+N-1} a^{\bar{j}}\right)\left(z \,\middle|\, \frac{k+N-1}{2}\right) \right) \tilde{S}_{\bar{j}}(z) :}$$

$$\left(\frac{1}{\beta} a^{\bar{j}}\right)(z|\alpha) = -\sum_{m\neq 0} \frac{a_m^{\bar{j}}}{[\beta m][m]} q^{-\alpha|m|} z^{-m}$$

$$+ \frac{1}{\beta}\left(Q_a^{\bar{j}} + a_0^{\bar{j}} \log z \right)$$

Scheening

$$\cdot\ \widehat{S}_i(z) = \frac{1}{(q-q^{-1})z} \sum_{j=i+1}^{N} \Big(:\exp\Big(-b_-^{ij}(q^{N-1-j}z) - (b+c)^{ij}(q^{N-j}z) + (b+c)^{i+1\,j}(q^{N-1-j}z)$$

$$+ \sum_{\ell=j+1}^{N} \big(b_-^{i+1\,\ell}(q^{N-\ell}z) - b_-^{i\ell}(q^{N-\ell-1}z)\big) + b_-^{i+1\,N+1}(z) - b_-^{i\,N+1}(q^{-1}z)\Big):$$

$$- :\exp\Big(-b_+^{ij}(q^{N-1-j}z) - (b+c)^{ij}(q^{N-j-2}z) + (b+c)^{i+1\,j}(q^{N-1-j}z)$$

$$+ \sum_{\ell=j+1}^{N} \big(b_-^{i+1\,\ell}(q^{N-\ell}z) - b_-^{i\ell}(q^{N-\ell-1}z)\big) + b_-^{i+1\,N+1}(z) - b_-^{i\,N+1}(q^{-1}z)\Big):\Big)$$

$$+ q :\exp\Big(b^{i\,N+1}(z) + b_+^{i+1\,N+1}(z) - b^{i+1\,N+1}(qz) \Big): ,$$

$$\cdot\ \widetilde{S}_N(z) = -q^{-1} :\exp\big(b^{N,N+1}(z)\big): .$$

Scheening

$$[\, a_{n,i} \,,\, S_{\bar{j}}(z) \,] = 0$$

$$[\, X_i^{+}(z_1) \,,\, S_{\bar{j}}(z_2) \,] = 0$$

$$[\, X_i^{-}(z_1) \,,\, S_{\bar{j}}(z_2) \,] = \frac{1}{(q - q^{-1}) z_1 z_2}$$

$$\times \left(\delta\left(q^{k+N-1} \frac{z_2}{z_1} \right) - \delta\left(q^{-k-N+1} \frac{z_2}{z_1} \right) \right) : \exp\left(-\left(\frac{1}{k+N-1} a^{\bar{j}} \right)\left(z_1 \middle| -\frac{k+N-1}{2} \right) \right) :$$

$$Q_{\bar{j}} = \int_0^{s\infty} S_{\bar{j}}(w)\, d_{q^{2(k+N-1)}} w$$ Jackson integral

$$\Rightarrow \quad [\, Q_{\bar{j}} \,,\, U_q \,] = 0$$

Screening

$$S_{\bar{i}}(z_1) S_{\bar{j}}(z_2) = \frac{\left[u_1 - u_2 + \frac{A_{\bar{i}\bar{j}}}{2}\right]_{k+N-1}}{\left[u_1 - u_2 + \frac{A_{\bar{i}\bar{j}}}{2}\right]_{k+N-1}} S_{\bar{j}}(z_2) S_{\bar{i}}(z_1)$$

$$(\bar{i}, \bar{j}) \neq (N, N)$$

$$\{S_N(z_1), S_N(z_2)\} = 0$$

Here $[u]_{k+N-1}$ is elliptic theta function.

$$X_{\bar{i}}^{\pm}(z_1) X_{\bar{j}}^{\pm}(z_2) = \frac{(q^{\pm A_{\bar{i}\bar{j}}} z_1 - z_2)}{(z_1 - q^{\pm A_{\bar{i}\bar{j}}} z_2)} X_{\bar{j}}^{\pm}(z_2) X_{\bar{i}}^{\pm}(z_1).$$

Elliptic Theta Function

- $$[u]_r = q^{\frac{u^2}{r}-u} \frac{\Theta_{q^{2r}}(q^{2u})}{(q^{2r};q^{2r})_\infty^3}$$

- $\Theta_p(z) = (z;p)_\infty (p/z;p)_\infty (p;p)_\infty$
- $(z;p)_\infty = \prod_{m=0}^{\infty} (1 - p^m z)$

Quasi-Periodicity

- $[u+r]_r = -[u]_r$
- $[u+r\tau]_r = -e^{2\pi\sqrt{-1}(u+\frac{\tau}{2})/r} [u]_r$ $\quad (\tau = \pi\sqrt{-1}/\log q)$

We have constructed the screening $Q_{\bar{j}}$ that commutes with $U_q(\widehat{sl}(N|1))$ for level $k \neq -N+1$.

Applications of Screenings

Example

(1) Background charge of the vertex operator is balanced by the screenings.

(2) Irreducible representation is constructed from Felder complex by the screenings.

Background Charge of Vertex Operator

$\boxed{\text{Vertex Operator}}$ $\Phi(z)$, $\Phi^*(z)$

- $\Phi(z) : \mathcal{F} \longrightarrow \mathcal{F}' \otimes V_{\alpha,z}$
- $\Phi^*(z) : \mathcal{F} \longrightarrow \mathcal{F}' \otimes V^{*S}_{\alpha,z}$
- $\mathcal{F}, \mathcal{F}'$: Wakimoto realization
- $V_{\alpha,z}$: Evaluation representation of Typical representation.
- $V^{*S}_{\alpha,z}$: Dual

Example

$\boxed{\text{Co-Product } \Delta \text{ of } U_q}$ $\qquad \Delta : U_q \longrightarrow U_q \otimes U_q$

$$\Delta(h_{\bar{i},m}) = h_{\bar{i},m} \otimes q^{\frac{km}{2}} + q^{\frac{3km}{2}} \otimes h_{\bar{i},m} \qquad \text{etc.}$$

$\boxed{\text{Intertwining Property}}$ $\qquad \Phi^*(z) : F \longrightarrow F' \otimes V_z^{*r}$

$$\Phi^*(z) \cdot x = \Delta(x) \cdot \Phi^*(z) \quad , \quad \Phi^*(z) = \sum_{\bar{j}} \Phi^*_{\bar{j}}(z) \otimes v^*_{\bar{j}}$$

$$\sum_{\bar{j}} \Phi^*_{\bar{j}}(z)\, h_{\bar{i},m} \otimes v^*_{\bar{j}} = \sum_{\bar{j}} \left(h_{\bar{i},m} \otimes q^{\frac{km}{2}} + q^{\frac{3km}{2}} \otimes h_{\bar{i},m} \right) \Phi^*_{\bar{j}}(z) \otimes v^*_{\bar{j}}$$

etc.

$\boxed{\text{Intertwining Property}}$

$$\Phi(z) = \sum_{\bar{j}=1}^{2^N} \Phi_{\bar{j}}(z) \otimes v_{\bar{j}}$$

$N=3$, $\alpha \neq 0,-1,-2$.

$\boxed{\Phi_{2^N}(z)}$

$$\Phi_1(z) = \frac{1}{\sqrt{[\alpha]}}[\Phi_2(z), f_3]_{q^{-\alpha}}$$

$$\Phi_2(z) = [\Phi_3(z), f_2]_q$$

$$\Phi_3(z) = [\Phi_4(z), f_1]_q \quad , \quad \Phi_3(z) = \frac{-1}{\sqrt{[\alpha+1]}}[\Phi_5(z), f_3]_{q^{-\alpha-1}}$$

$$\Phi_4(z) = \frac{-1}{\sqrt{[\alpha+1]}}[\Phi_6(z), f_3]_{q^{-\alpha-1}}$$

$$\widehat{sl}(3|1)$$

$$\Phi_5(z) = [\Phi_6(z), f_1]_q$$

$$\Phi_6(z) = [\Phi_7(z), f_2]_q$$

$$\Phi_7(z) = \frac{1}{\sqrt{[\alpha+2]}}[\Phi_8(z), f_3]_{q^{-\alpha-2}}$$

$\boxed{\text{Typical Representation}}$ of classical

$\Longleftrightarrow$

Weyl's Character Formula holds.

$$\mathrm{ch}\, V(\lambda) = \sum_{\sigma} \operatorname{sgn}\sigma\, e^{\sigma(\lambda+\rho)} \Big/ \sum_{\sigma} \operatorname{sgn}\sigma\, e^{\sigma(\rho)}.$$

Example.

$\boxed{\text{Non-super}}$ finite dim. irreducible representation

$\boxed{\text{Super}}$ Ex. $sl(N|1)$

Smallest dimension of typical rep. 2^N

dimension of basic rep. $N+1$

$\boxed{\text{Evaluation of Typical}}$ $U_q(\widehat{sl}(3|1))$ $V_{\alpha,z}$ $\dim = 8$

$$h_{1,m} = \frac{[m]}{m}(q^{\alpha+2}z)^m(q^{-m}E_{33} - q^m E_{44} + q^{-m}E_{55} - q^m E_{66}),$$

$$h_{2,m} = \frac{[m]}{m}(q^{\alpha+2}z)^m(q^{-2m}E_{22} - E_{33} + E_{66} - q^{2m}E_{77}),$$

$$h_{3,m} = \frac{1}{m}z^m\Big([\alpha m](E_{11}+E_{22}) + [(\alpha+1)m]q^m(E_{33}+E_{44}+E_{55}+E_{66}) + [(\alpha+2)m]q^{2m}(E_{77}+E_{88})\Big),$$

$$x^+_{1,n} = (q^{\alpha+2}z)^n(E_{34}+E_{56}),$$

$$x^+_{2,n} = (q^{\alpha+2}z)^n(q^{-n}E_{23}+q^n E_{67}),$$

$$x^+_{3,n} = (q^{\alpha+2}z)^n\left(\sqrt{[\alpha]}\,q^{-2n}E_{12} - \sqrt{[\alpha+1]}(E_{35}+E_{46}) + \sqrt{[\alpha+2]}\,q^{2n}E_{78}\right),$$

$$x^-_{1,n} = (q^{\alpha+2}z)^n(E_{43}+E_{65}),$$

$$x^-_{2,n} = (q^{\alpha+2}z)^n(q^{-n}E_{32}+q^n E_{76}),$$

$$x^-_{3,n} = (q^{\alpha+2}z)^n\left(\sqrt{[\alpha]}\,q^{-2n}E_{21} - \sqrt{[\alpha+1]}(E_{53}+E_{64}) + \sqrt{[\alpha+2]}\,q^{2n}E_{87}\right).$$

Vertex Operator

$l_a = (l_a^1, l_a^2, \cdots, l_a^N) \in \mathbb{C}^N, \quad \beta \in \mathbb{C}$

$$\boxed{\phi^{l_a}(z|\beta) = :\exp\left(\sum_{i,j=1}^{N}\left(\frac{l_a^i}{k+N-1}\frac{\mathrm{Min}(i,j)}{N-1}\frac{N-1-\mathrm{Max}(i,j)}{1}a^j\right)(z|\beta)\right):}$$

- $\left(\frac{\alpha_1}{\beta_1}\frac{\alpha_2}{\beta_2}\cdots\frac{\alpha_r}{\beta_r}a^i\right)(z|\alpha)$

$$= -\sum_{m\neq 0}\frac{[\alpha_1 m]\cdots[\alpha_r m]}{[\beta_1 m]\cdots[\beta_r m]}\frac{a_m^i}{[m]}q^{-\alpha|m|}z^{-m} + \frac{\alpha_1\alpha_2\cdots\alpha_r}{\beta_1\beta_2\cdots\beta_r}(Q_a^i + a_0^i\log z).$$

- $[h_{i,m}, \phi^{l_a}(z|\beta)] = \frac{1}{m}[l_a^i m]\, q^{-(\beta+\frac{N-1}{2})|m|} z^m\, \phi^{l_a}(z|\beta)$
- $[X_i^+(z_1), \phi^{l_a}(z_2|\beta)] = 0$
- $(z_1 - q^{l_a^i}z_2)\, X_i^-(z_1)\, \phi^{l_a}(z_2| -\frac{k+N-1}{2})$

$$= (q^{l_a^i}z_1 - z_2)\, \phi^{l_a}(z_2| -\frac{k+N-1}{2})\, X_i^-(z_1) \qquad (1\leq i\leq N)$$

Vertex Operator

V : Typical Representation with Weight $\lambda = \sum_{j=1}^{N} l_a^{j} \Lambda_j$.

$$\boxed{\begin{aligned} \Phi_1^*(z) &= \eta_0 \xi_0 \, \phi^{l_a}\left(q^k z \,\middle|\, -\frac{k+N-1}{2}\right) \eta_0 \xi_0 \\ \Phi_{N_V}(z) &= \eta_0 \xi_0 \, \phi^{-l_a-N+1}\left(q^{k+N-1} z \,\middle|\, -\frac{k+N-1}{2}\right) \eta_0 \xi_0 \end{aligned}}$$

$$\Phi(z) = \sum_{j=1}^{N_V} \Phi_j(z) \otimes v_j , \quad \Phi^*(z) = \sum_{j=1}^{N_V} \Phi_j^*(z) \otimes v_j^*$$

$$\left(N_V = \dim V = 2^N \prod_{1 \le i < j \le N-1} \frac{l_a^{i} + \cdots + l_a^{j} + j - i + 1}{j - i + 1} \quad \text{[Kac1978]} \right)$$

Other components $\Phi_j(z)$, $\Phi_j^*(z)$ are determined by

$$\Phi(z) x = \Delta(x) \Phi(z) \quad \text{etc.}$$

Vertex Operator

Conjecture Our bosonizations $\Phi^*(z)$, $\Phi(z)$ satisfy the intertwing properties.

$$\Phi^*(z) : \mathcal{F}(p_a) \longrightarrow \mathcal{F}(p_a + l_a) \otimes V_z^{*S}$$

$$\Phi(z) : \mathcal{F}(p_a) \longrightarrow \mathcal{F}(p_a - l_a - N + 1) \otimes V_z$$

We have checked it in some cases for $N = 2, 3, 4$, by using Gelfand-Zetlin basis.

$$\boxed{\text{Correlation Function}}$$

$$\mathrm{Tr}_{\mathcal{F}(p_a)}\Big(q^{L_0}\, \Phi^*_{\bar{i}_1}(w_1)\, \Phi^*_{\bar{i}_2}(w_2) \cdots \Phi^*_{\bar{i}_m}(w_m)$$

$$\times\, \Phi_{\bar{j}_1}(z_1)\, \Phi_{\bar{j}_2}(z_2) \cdots \Phi_{\bar{j}_n}(z_n) \Big)$$

$\boxed{\text{Correlation Function}}$

- $$\mathrm{Tr}_{\mathcal{F}(p_a)}\left(q^{L_0}\, \Phi^*_{\bar{i}_1}(w_1)\, \Phi^*_{\bar{i}_2}(w_2) \cdots \Phi^*_{\bar{i}_m}(w_m) \times \Phi_{\bar{j}_1}(z_1)\, \Phi_{\bar{j}_2}(z_2) \cdots \Phi_{\bar{j}_n}(z_n) \right) = 0 \quad !?$$

- Product $\Phi^*_{\bar{i}_1}(w_1) \cdots \Phi^*_{\bar{i}_m}(w_m)\, \Phi_{\bar{j}_1}(z_1) \cdots \Phi_{\bar{j}_n}(z_n)$ doesn't preserve the space $\mathcal{F}(p_a)$.

Background Charge

$$\Phi(z) : \mathcal{F}(p_a) \longrightarrow \mathcal{F}(p_a + l_a) \otimes V_z$$

$$\Downarrow$$

$$\widetilde{\Phi}(z) : \mathcal{F}(p_a) \longrightarrow \mathcal{F}(p_a + l_a + \hat{x}) \otimes V_z$$

(1) Preserve the intertwining relation

(2) Change the space $\mathcal{F}(p_a + l_a)$

$\Longrightarrow$ Schreening Q_j

$\boxed{\text{Background Charge}}$

$Q_{\bar{J}}$: screening

- $[Q_{\bar{J}}, U_q] = 0$
- $Q_{\bar{J}}$ change the space $\mathcal{I}(P_a)$

Vertex Operator

$$\widetilde{\Phi}^*_1(z) = :\eta_0\xi_0\, Q_1^{x_1} Q_2^{x_2} \cdots Q_N^{x_N}\, \phi^{l_a}(q^k z \,|\, -\tfrac{k+N-1}{2})\, \eta_0\xi_0 :$$

$$\widetilde{\Phi}_{N_v}(z) = :\eta_0\xi_0\, Q_1^{x_1} Q_2^{x_2} \cdots Q_N^{x_N}\, \phi^{-l_a-N+1}(q^{k+N-1} z \,|\, -\tfrac{k+N-1}{2})\, \eta_0\xi_0 :$$

Conjecture $\hat{x} = (\hat{x}_1, \hat{x}_2, \cdots, \hat{x}_N)$, $\hat{x}_i = \sum_{j=1}^{N} A_{ij} x_j$

$$\widetilde{\Phi}(z) : \mathcal{F}(p_a) \longrightarrow \mathcal{F}(p_a + l_a + \hat{x}) \otimes V_z$$

$$\widetilde{\Phi}^*(z) : \mathcal{F}(p_a) \longrightarrow \mathcal{F}(p_a - l_a - N + 1 + \hat{x}) \otimes V_z^*$$

We have checked it in some cases for $N = 2, 3, 4$.

$\boxed{\text{Correlation Function}}$

$$\mathrm{Tr}_{\mathcal{F}(p_a)}\Big(q^{L_0}\, \tilde{\Phi}^*_{\bar{i}_1}(w_1)\, \tilde{\Phi}^*_{\bar{i}_2}(w_2) \cdots \tilde{\Phi}^*_{\bar{i}_m}(w_m)$$

$$\times\, \tilde{\Phi}_{\bar{j}_1}(z_1)\, \tilde{\Phi}_{\bar{j}_2}(z_2) \cdots \tilde{\Phi}_{\bar{j}_n}(z_n) \Big) \neq 0 .$$

· We used screening $Q_{\bar{j}}$ to get non-zero trace.

" We modify background charge of vertex operator "

Summary

- We constructed the bosonization of $U_q(\widehat{sl}(N|1))$ for an arbitrary level k.
- We constructed the screening Q_j that commutes with $U_q(\widehat{sl}(N|1))$.
- We propose the bosonization of vertex operators among Wakimoto realization and typical representation by using screenings.

Elliptic Algebra U_{qp}

Quantum Group $U_q(\widehat{sl_2})$

6 - vertex model

Trigonometric function

Elliptic Quantum Group $\mathcal{B}_{q\lambda}(\widehat{sl_2})$

ABF - face model

Elliptic function

$\boxed{\text{Elliptic Algebra}}$ $U_{qp}(\widehat{sl_2})$

Drinfeld current $x^{\pm}(z)$ of $U_q(\widehat{sl_2})$

$$\Downarrow \text{ elliptic deformation}$$

Current $E(z), F(z)$ of $U_{qp}(\widehat{sl_2})$

- $U_{qp}(\widehat{sl_2})$ plays an important role in construction of Vertex Operator, L-operator of $\mathcal{B}_{q\lambda}(\widehat{sl_2})$.

$\widehat{sl}(M|N)$ analogue of ABF-face model

Elliptic Algebra $U_{q,p}(\widehat{sl}(M|N))$ and its Bosonizations

$\boxed{\text{Elliptic Superalgebra}}$ $U_{q,p}(\widehat{sl}(M|N))$

$p = q^{2r}, \quad r, \quad r^* = r - k \quad (\mathrm{Re}(r), \mathrm{Re}(r^*) > 0)$

$$\boxed{\begin{array}{l} \cdot\ U_i^+(z) = \exp\left(\sum_{m>0} \frac{q^{rm}}{[rm]} B_{i,-m} z^m \right) \\ \hline \cdot\ U_i^-(z) = \exp\left(-\sum_{m>0} \frac{q^{r^*m}}{[r^*m]} B_{i,m} z^{-m} \right) \end{array}}$$

Dressing Operator

$$\cdot\ B_{j,m} = \begin{cases} \dfrac{[r^*m]}{[rm]} a_{j,m} & (m>0) \\ q^{k|m|} a_{j,m} & (m<0) \end{cases}$$

[Kojima, JPA44, 485205 (23pp) (2011)]

$\boxed{\text{Elliptic Superalgebra}}$ $U_{q,p}(\widehat{sl}(M|N))$

$$\boxed{\begin{aligned}
&\cdot\ E_j(z) = U_j^+(z)\, X_j^+(z)\, e^{2Q_j} z^{-\frac{1}{r^*}P_j}, \\
&\cdot\ F_j(z) = X_j^-(z)\, U_j^-(z)\, z^{\frac{1}{r}(P_j+h_j)}, \qquad (1 \le j \le M+N-1) \\
&\cdot\ H_j^{\pm}(z) = H_j\left(q^{\pm(r-\frac{k}{2})} z\right), \\
&\cdot\ H_j(z) = :\exp\left(-\sum_{m\neq 0} \frac{B_{j,m}}{[r^* m]} z^{-m}\right): e^{2Q_j} z^{-\frac{1}{rr^*}P_j + \frac{1}{r}h_j}.
\end{aligned}}$$

$$[P_i, Q_j] = -\frac{A_{i,j}}{2} \qquad (1 \le i, j \le M+N-1)$$

Elliptic Superalgebra $U_{q,p}(\widehat{sl}(M|N))$

$r^* = r - k$

- $\left[u_1 - u_2 - \frac{A_{\bar{i}\bar{j}}}{2}\right]_{r^*} E_{\bar{i}}(z_1) E_{\bar{j}}(z_2) = \left[u_1 - u_2 + \frac{A_{\bar{i}\bar{j}}}{2}\right]_{r^*} E_{\bar{j}}(z_2) E_{\bar{i}}(z_1)$,

- $\left[u_1 - u_2 + \frac{A_{\bar{i}\bar{j}}}{2}\right]_{r} F_{\bar{i}}(z_1) F_{\bar{j}}(z_2) = \left[u_1 - u_2 - \frac{A_{\bar{i}\bar{j}}}{2}\right]_{r} F_{\bar{j}}(z_2) F_{\bar{i}}(z_1)$,

- $\left[E_{\bar{i}}(z_1), F_{\bar{j}}(z_2)\right] = \frac{\delta_{\bar{i}\bar{j}}}{(q - q^{-1}) z_1 z_2} \times$ $\quad (\bar{i}, \bar{j}) \neq (M, N)$

  $\times \left(\delta\left(q^{-k} \frac{z_1}{z_2}\right) H_{\bar{i}}(q^{r} z_2) - \delta\left(q^{k} \frac{z_1}{z_2}\right) H_{\bar{i}}(q^{-r} z_2) \right)$,

- $\left\{ E_M(z_1), F_M(z_2) \right\} = \frac{1}{(q - q^{-1}) z_1 z_2} \times$

  $\times \left(\delta\left(q^{-k} \frac{z_1}{z_2}\right) H_M(q^{r} z_2) - \delta\left(q^{k} \frac{z_1}{z_2}\right) H_M(q^{-r} z_2) \right)$,

- $\left\{ E_M(z_1), E_M(z_2) \right\} = 0$, $\left\{ F_M(z_1), F_M(z_2) \right\} = 0$.

etc.

Bosonization of U_{qp}

Using dressing operators $U_i^{\pm}(z)$

and

bosonizations of $U_q(\widehat{sl}(N|1))$

we have bosonization of $U_{qp}(\widehat{sl}(N|1))$.

$$F(p_a) = \bigoplus_{\substack{p_b^{ij} = -p_c^{ij} \in \mathbb{Z} \ (1 \le i < j \le N) \\ p_b^{i\,N+1} \in \mathbb{Z} \ (1 \le i \le N)}} F(p_a, p_b, p_c)$$

$\boxed{\text{Screening}}$ $Q_{\bar{j}}$, $U_{qp}(\widehat{sl}(N|1))$

- Screening of elliptic algebra $U_{qp}(\widehat{sl}(N|1))$ is exactlly the same as those of $U_q(\widehat{sl}(N|1))$.

We understand it from bosonization of $U_q(\widehat{sl}(N|1))$.

$$
\boxed{
\begin{aligned}
&[a_{m,\bar{i}}, S_{\bar{j}}(z)] = 0 \\
&[E_{\bar{i}}(z_1), S_{\bar{j}}(z_2)] = 0 \\
&[F_{\bar{i}}(z_1), S_{\bar{j}}(z_2)] = \frac{1}{(q-q^{-1})z_1 z_2} \\
&\times \left(\delta\left(q^{k+N-1}\frac{z_2}{z_1}\right) - \delta\left(q^{-k-N+1}\frac{z_2}{z_1}\right) \right) :\exp(\text{bosons}(z_1)):
\end{aligned}
}
$$

$\boxed{\text{Screening}}$

$k \neq -N+1$

$$\boxed{S_{\bar{j}}(z) = :\exp\left(-\left(\frac{1}{k+N-1} a^{\bar{j}}\right)\left(z \,\middle|\, \frac{k+N-1}{2}\right)\right)\tilde{S}_{\bar{j}}(z):}$$

$$\left(\frac{1}{\beta} a^{\bar{j}}\right)(z|\alpha) = -\sum_{m \neq 0} \frac{a_m^{\bar{j}}}{[\beta m][m]} q^{-\alpha|m|} z^{-m}$$

$$+ \frac{1}{\beta}\left(Q_a^{\bar{j}} + a_0^{\bar{j}} \log z\right)$$

Scheening

$$\cdot\ \widehat{S}_i(z) = \frac{1}{(q-q^{-1})z} \sum_{j=i+1}^{N} \Big(:\exp\Big(-b_-^{ij}(q^{N-1-j}z) - (b+c)^{ij}(q^{N-j}z) + (b+c)^{i+1\,j}(q^{N-1-j}z)$$

$$+ \sum_{\ell=j+1}^{N} \big(b_-^{i+1\,\ell}(q^{N-\ell}z) - b_-^{i\ell}(q^{N-\ell-1}z)\big) + b_-^{i+1\,N+1}(z) - b_-^{i\,N+1}(q^{-1}z)\Big):$$

$$- :\exp\Big(-b_+^{ij}(q^{N-1-j}z) - (b+c)^{ij}(q^{N-j-2}z) + (b+c)^{i+1\,j}(q^{N-1-j}z)$$

$$+ \sum_{\ell=j+1}^{N} \big(b_-^{i+1\,\ell}(q^{N-\ell}z) - b_-^{i\ell}(q^{N-\ell-1}z)\big) + b_-^{i+1\,N+1}(z) - b_-^{i\,N+1}(q^{-1}z)\Big):\Big)$$

$$+ q :\exp\Big(b^{i\,N+1}(z) + b_+^{i+1\,N+1}(z) - b^{i+1\,N+1}(qz)\Big): ,$$

$$\cdot\ \widetilde{S}_N(z) = -q^{-1} :\exp\big(b^{N,N+1}(z)\big): .$$

$\boxed{\text{Scheening}}$ $\quad Q_{\bar{j}} \quad (1 \leq \bar{j} \leq N) \qquad k \neq -N+1.$

$$\boxed{Q_{\bar{j}} = \int_0^{s\infty} S_{\bar{j}}(z)\, d_p z \qquad (p = q^{2(k+N-1)})}$$

$$\boxed{\begin{aligned} &[\, Q_{\bar{j}},\ U_{qp}(\widehat{sl}(N|1))\,] = 0 \\ &[\, Q_{\bar{j}},\ \eta_0 \xi_0 \,] = 0 \end{aligned}}$$

$\boxed{\text{Wakimoto realization}}$

- $[\eta_0 \xi_0, U_{qp}] = 0$ — ξ-η system
- $\mathcal{F}(p_a) = \eta_0 \xi_0 F(p_a)$ — Wakimoto realization

$\boxed{\text{Vertex operator}}$

Open

Summary

- We introduced elliptic algebra $U_{q,p}(\widehat{sl}(M|N))$.
- We give a bosonization of $U_{q,p}(\widehat{sl}(N|1))$ for an arbitrary level k.
- Screenings Q_i of $U_{q,p}(\widehat{sl}(N|1))$ coincides with those of $U_q(\widehat{sl}(N|1))$.

Next step is construction of Vertex Operator.

Summary

- We construct a bosonization of superalgebra $U_q(\widehat{sl}(N|1))$ for an arbitrary level k.
- We propose a bosonization of vertex operator of $U_q(\widehat{sl}(N|1))$.
- We introduce elliptic algebra $U_{q,p}(\widehat{sl}(N|M))$ and construct a bosonization for $M=1$.

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