Institute of Mathematics of National Academy of Sciences of Ukraine
3 Tereshchenkivs'ka Str., Kyiv, Ukraine

Classification of shape-invariant Schrödinger equations

Yuri Karadzhov
yuri.karadzhov@gmail.com

9 - 19 September 2012
7th MATHEMATICAL PHYSICS MEETING:
Summer School and Conference on Modern Mathematical Physics
Belgrade, Serbia

Three types of mathematical problems

All mathematical problems can be generally divided into next three types

- **Proof**
- **Method searching**
- **Classification**

Classification problems aim to find among given class of problems all subclasses, that can be solved by specific method.

Classification $\supset$ *Method searching* $\supset$ *Proof*

Steps to solve classification problem

- Choose the method
- Fix the class
- Define equivalence transformations
- Search for nonequivalent subclasses of the given class, that can be solved by the given method
- Find additional equivalence transformations

Classification of equations

+ Classification of equations that admit Lie symmetries
+ Classification of equations that admit potential symmetries
+ Classification of equations that admit conditional symmetries
+ Classification of superintegrable equations

— Classification of shape-invariant equations

Shape-invariant scalar superpotentials

1. $W = \mu x, \quad$ (harmonic oscilator)
2. $W_k = \mu x - \frac{k}{x}, \quad$ (3D oscilator)
3. $W_k = \lambda k \tan \lambda x + \mu \sec \lambda x, \quad$ (Scarf 1)
4. $W_k = \lambda k \tanh \lambda x + \mu \,\mathrm{sech}\, \lambda x, \quad$ (Scarf 2)
5. $W_k = \lambda k \coth \lambda x + \mu \,\mathrm{csch}\, \lambda x, \quad$ (Pöschl-Teller)
6. $W_k = k - \mu \exp(-x), \quad$ (Morse)
7. $W_k = -\frac{k}{x} + \frac{\omega}{k}, \quad$ (Coulomb)
8. $W_k = \lambda k \tan \lambda x + \frac{\omega}{k}, \quad$ (Rosen-Morse 1)
9. $W_k = \lambda k \tanh \lambda x + \frac{\omega}{k}, \quad$ (Rosen-Morse 2)
10. $W_k = -\lambda k \coth \lambda x + \frac{\omega}{k}, \quad$ (Eckart)

Pron'ko-Stroganov problem

Consider the spectral problem for radial functions

$$H_k \psi = E_k \psi, \tag{1}$$

where H_k is a Hamiltonian with a matrix potential, E_k and ψ are its eigenvalue and eigenfunction correspondingly, moreover, ψ is a two-component spinor. Up to normalization of the radial variable x the Hamiltonian H_k can be represented as

$$H_k = -\frac{\partial^2}{\partial x^2} + k(k - \sigma_3)\frac{1}{x^2} + \sigma_1 \frac{1}{x}, \tag{2}$$

where σ_1 and σ_3 are Pauli matrices and k is a natural number. In addition, solutions of equation (1) must be normalizable and vanish at the boundary of the interval $(0, \infty)$.

Pron'ko-Stroganov problem

Hamiltonian H_k can be factorized as

$$H_k = a_k^+ a_k^- + c_k, \tag{3}$$

where

$$a_k^- = \frac{\partial}{\partial x} + W_k, \quad a_k^+ = -\frac{\partial}{\partial x} + W_k, \quad c_k = -\frac{1}{(2k+1)^2}$$

and W is a *matrix superpotential*

$$W_k = \frac{1}{2x}\sigma_3 - \frac{1}{2k+1}\sigma_1 - \frac{2k+1}{2x}. \tag{4}$$

Another nice property of Hamiltonian H_k is that its superpartner H_k^+ is equal to H_{k+1}, namely

$$H_k^+ = a_k^- a_k^+ + c_k = -\frac{\partial^2}{\partial x^2} + (k+1)(k+1-\sigma_3)\frac{1}{x^2} + \sigma_1\frac{1}{x} = H_{k+1}$$

Thus equation (1) admits supersymmetry with shape invariance and can be solved by algebraic methods.

Problem statement

- **Equation**

$$H_k\psi = -\frac{\partial^2\psi}{\partial x^2} + V_k(x)\psi = E_k\psi, \tag{5}$$

where $V_k(x)$ is $n \times n$ dimensional matrix potential.

- **Factorization**

$$H_k = a_k^+ a_k^- + c_k, \quad H_k^+ = a_k^- a_k^+ + c_k, \tag{6}$$

where

$$a_k^+ = -\frac{\partial}{\partial x} + W_k, \quad a_k^- = \frac{\partial}{\partial x} + W_k, \tag{7}$$

c_k is a scalar function of k, that vanishes with a corresponding member in the Hamiltonian. $W_k(x)$ is *matrix superpotential*.

- **Shape-invariance**

$$H_k^+ = H_{F_k}. \tag{8}$$

Shape-invariance condition

Under invertible transformation of variables

$$k \to \alpha(k) \tag{9}$$

the function $F_k = F(k)$ changes by a similar transformation

$$F(k) \to \alpha F \alpha^{-1}(k) = \alpha(F(\alpha^{-1}(k))). \tag{10}$$

Searching for such transformation, that would change function $F(k)$ to unit shift, we get the equation

$$F(\alpha^{-1}(k)) = \alpha^{-1}(k+1). \tag{11}$$

The above equation is known as Abel functional equation.
Let X be $\mathbb{R}$ or $\mathbb{R}^+$. It is proved that

(C1) *if* $F : X \to \mathbb{R}$ *is an injective function such that for every compact set* $K \subset X$ *there exists* $p \in \mathbb{N}$ *such that* $\forall n, m \in \mathbb{N}^0,\ |n-m| \geq p$:

$$F^n(K) \cap F^m(K) = \emptyset$$

then there exists a solution to the Abel functional equation.

So if the above condition is fulfilled, F_k can be transformed into unit shift.

Class of superpotentials

- **Problem** - to find all shape-invariant potentials, which accept factorization (6)

$$W_k^2 + W_k' = W_{k+1}^2 - W_{k+1}' + C_k, \tag{12}$$

where $C_k = c_{k+1} - c_k$.

- **Superpotentials of the special form**

$$W_k = kQ + P + \frac{1}{k}R, \tag{13}$$

where P, Q, R are $n \times n$ Hermitian matrices depending on x.

- **Irreducibility** - matrices P, Q and R cannot be simultaneously transformed to a block diagonal form since if such (unitary) transformation is admissible, the related superpotentials are completely reducible.

Equivalence transformations

In the set Ω_n of all superpotentials of dimension n, which is the set of all hermitian matrices of dimension n, that depends on variable x and parameter k, introduce an equivalence transformation:

We say, that two superpotentials $\tilde{W}_k, W_k \in \Omega_n$ are equivalent $\tilde{W}_k \sim W_k$, if there exists unitary matrix U, which doesn't depend on variable x such, that

$$\tilde{W}_k = U W_k U^\dagger \tag{14}$$

Beside that, consider following two potentials to be equivalent

$$W_k(x) \sim W_k(x+\gamma), \tag{15}$$

where $\gamma \in \mathbb{R}$ — is an arbitrary constant.

The determining equations

The system of determining equations:

1. $Q' = Q^2 + \nu$
2. $P' = \frac{1}{2}\{P, Q\} + \mu$
3. $R' = 0$
4. $R^2 = \omega^2$
5. $\{P, R\} + \varkappa = 0$
6. $C_k = 2\mu + (2k+1)\nu - \frac{\varkappa}{k(k+1)} + \frac{(2k+1)\omega^2}{k^2(k+1)^2}$

where $\nu, \mu, \omega, \varkappa$ are arbitrary real constants.

Irreducible special cases

Cases, when one of the matrices P, Q or R is proportional to the unit matrix or equal to zero matrix are special.

Irreducible special cases

1. $Q = q(x)I$
2. $R = 0$
3. $P = 0$
4. P, Q, R are not proportional to zero or unit matrices

The case when two or three of these matrices are equal to zero or proportional to the unit matrix is obviously reducible. Because in this case they can always be diagonalized with some unitary transformation which doesn't depend on variable x (thought can depend on parameter k). The exception is scalar case, when matrices has dimension equal to one and can't be divided into blocks.

$Q = q(x)I$

- **Reducibility** - if $n > 2$ then up to unitary transformation $W_k(x)$ is a direct sum of 1×1 and 2×2 irreducible superpotentials.
- **Solvability** - the problem can be completely solved in Pauli matrix basis:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{16}$$

Shape-invariant matrix superpotentials

1. $W_k = \left((2\mu+1)\,\sigma_3 - 2k - 1\right)\frac{1}{2x} + \frac{\omega}{2k+1}\sigma_1, \quad \mu > -\frac{1}{2},$
2. $W_k = \lambda\left(-k + \exp(-\lambda x)\sigma_1 - \frac{\omega}{k}\sigma_3\right),$
3. $W_k = \lambda\left(k\tan\lambda x + \mu\sec\lambda x\sigma_3 + \frac{\omega}{k}\sigma_1\right), \quad \mu > 0,$
4. $W_k = \lambda\left(-k\coth\lambda x + \mu\,\mathrm{csch}\,\lambda x\sigma_3 - \frac{\omega}{k}\sigma_1\right), \quad \mu < 0,$
5. $W_k = \lambda\left(-k\tanh\lambda x + \mu\,\mathrm{sech}\,\lambda x\sigma_1 - \frac{\omega}{k}\sigma_3\right), \quad \mu > 0,$

where $\omega > 0$.

- Superpotentials are defined up to unitary transformation and shifts of variable x and parameter k.
- If $\mu = 0$ and $\omega = 1$ then 1 defines well known superpotential for Pron'ko-Stroganov problem.
- Corresponding potentials
  $V_k = W_k^2 - W_k' + c_k.$

Dual shape-invariance

- **Inverse problem** - to find possible superpotentials corresponding to given potentials.
- **Invariance** - potentials corresponding to 1, 3, 4 are invariant with respect to the simultaneous change

$$\mu \rightarrow k - \frac{1}{2}, \quad k \rightarrow \mu + \frac{1}{2}. \tag{17}$$

  In addition, there exist another transformations of μ and k but they lead to the same results.
- **Dual shape-invariance** - superpotentials 1, 3, 4, should be considered together with superpotentials which can be obtained using the change (17).
  Thus corresponding potentials admit a dual supersymmetry, i.e., superpartners for these potentials can be obtained either by shifts of k or by shifts of μ while simultaneous shifts are forbidden. We call this phenomena *dual shape invariance*.

$R = 0$

The determining equations have the form

1. $Q' = Q^2 + \nu$
2. $P' = \frac{1}{2}\{P, Q\} + \mu$
3. $C_k = 2\mu + (2k+1)\nu$

- Matrix Q can be diagonalized with unitary transformation, which does not depend on variable x.
- The second equation can be solved element-wise.
- Additional equivalence transformation $k \to k + \beta$, $P \to P + \beta Q$.
- Additional equivalence transformations for some members of the class.

$P = 0$

The system of determining equations has the form

1. $Q' = Q^2 + \nu,$
2. $R' = 0, \quad R^2 = \omega^2,$
3. $\delta_k = (2k+1)\nu + \frac{(2k+1)\omega^2}{k^2(k+1)^2}.$

- Matrix Q can be diagonalized with unitary transformation, which does not depend on the variable x.
- Matrix R has the form $R = U\tilde{R}U^\dagger$, where

$$\tilde{R} = \omega \begin{pmatrix} I_{m\times m} & 0_{m\times s} \\ 0_{s\times m} & -I_{s\times s} \end{pmatrix}, \quad m + s = n,$$

U — special unitary matrix, which is not block-diagonal.

P, Q, R are not proportional to unit or zero matrices

- Matrix Q can be diagonalized with unitary transformation which, does not depend on variable x.
- The equation for matrix P can be solved element-wise.
- Matrix R — constant matrix, with the square proportional to the unit one.
- Consistency condition

$$\mu = 0, \quad \varkappa = 0$$

- Algebraic condition

$$\{P, R\} = 0$$

Ground and excited states

Schrödinger equation

$$H_k\psi \equiv (a_k^+ a_k^- + c_k)\psi = E_k\psi$$

can be integrated using an algebraic methods:

- **Ground state** - proportional to the square integrable solutions of the first order equation

$$a_k^- \psi_k^0(x) \equiv \left(\frac{\partial}{\partial x} + W_k\right)\psi_k^0(x) = 0. \tag{18}$$

  with energy $E_k^0 = c_k$.
- **Excited states** - solutions which correspond to n^{th} excited state can be represented as

$$\psi_k^n(x) = a_k^+ a_{k+1}^+ \cdots a_{k+n-1}^+ \psi_{k+n}^0(x). \tag{19}$$

  The corresponding eigenvalue is $E_k^n = E_k^0 + \sum_{i=0}^{n-1} C_{k+i}$.
- **Dual shape-invariance** - it is necessary to repeat the steps enumerated above using additional superpotentials.

Ground and excited states

- **Superpotential**

$$W_k = \lambda\left(-k + \exp(-\lambda x)\sigma_1 - \frac{\omega}{k}\sigma_3\right) \tag{20}$$

- **Ground state**

$$\psi_k^0(x) = \begin{pmatrix} y^{\frac{1}{2}-k} K_{|\nu|}(y) \\ -y^{\frac{1}{2}-k} K_{|\nu-1|}(y), \end{pmatrix}, \tag{21}$$

where $y = \exp(-\lambda x)$ and $\nu = \omega/k + 1/2$.

- **Energy spectrum**

$$E = -\lambda^2\left(N^2 + \frac{\omega^2}{N^2}\right), \quad N = n + k \tag{22}$$

- **Integrability condition**

$$k < 0, \quad k^2 > \omega \tag{23}$$

Ground and excited states

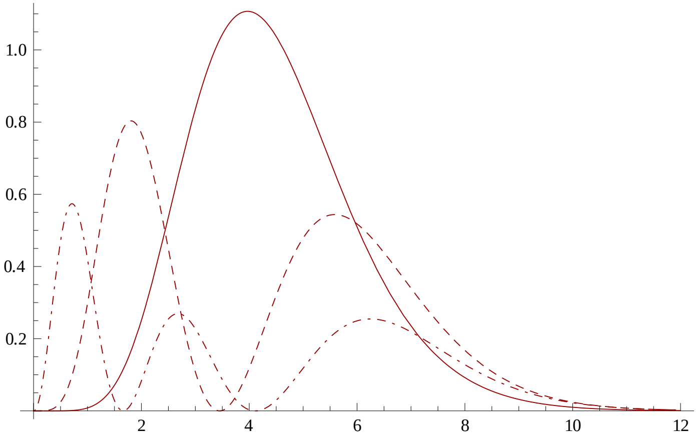

Figure: Probability densities $(\psi_k^n)^\dagger \psi_k^n$ versus x for superpotential 2 with $\lambda = 1, k = -4, \omega = 2$: $n = 0$ (continues), $n = 1$ (dashed) and $n = 2$ (dashed and pointed)

References

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Thank you!