Unitary similarity. New criteria

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Operators R and S are called unitary similar, if there is a unitary operator U such that $S = U^*RU$.

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Invariants under unitary similarity:

- norm
- trace
- spectrum
- numerical range

Specht's theorem (1940)

Operators R and S on a finite-dimensional Hilbert space H are unitarily similar if and only if

$$\text{trace}\ \omega(R, R^*) = \text{trace}\ \omega(S, S^*)$$

for every word $\omega(s, t)$ in two noncommuting variables.

Finite-dimensional Hilbert space

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Pappacena's restriction (1997)

Operators R and S on an n-dimensional Hilbert space H are unitarily similar if and only if $\mathrm{trace}\ \omega(R, R^*) = \mathrm{trace}\ \omega(S, S^*)$ for every word $\omega(s, t)$ in two noncommuting variables whose length is at most

$$n\sqrt{\frac{2n^2}{n-1} + \frac{1}{4}} + \frac{n}{2} - 2$$

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This theorem is extended to compact operators in trace- and Schmidt classes on infinite-dimensional Hilbert spaces and cannot be extended to all compact operators.

For each matrix polynomial

$$p(x) = A_0 + A_1 x + \cdots + A_t x^t \in \mathbb{C}^{k\times k}[x],$$

whose coefficients A_i are $k \times k$ matrices, we define its **value** at an operator $R \in B(H)$ as follows:

$$p(R) := A_0 \otimes I + A_1 \otimes R + \cdots + A_t \otimes R^t \in \mathbb{C}^k \otimes H,$$

where I is the identity operator and

$$\begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix} \otimes R := \begin{bmatrix} a_{11}R & \dots & a_{1n}R \\ \vdots & & \vdots \\ a_{m1}R & \dots & a_{mn}R \end{bmatrix}.$$

Arveson's theorem (1972)

Two irreducible operators R and S acting on a Hilbert space, such that the orders $n(R)$ and $n(S)$ are defined and equal, are unitarily similar if and only if

$$\|f(R)\|_{op} = \|f(S)\|_{op} \text{ for all } f \in \mathbb{C}^{k\times k}[x] \text{ and } k = 1, 2, \ldots, n(R), \tag{1}$$

where $\|\cdot\|_{op}$ is the operator norm.

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where $\|\cdot\|_{op}$ is the operator norm.

Question For which classes of operators it suffices to check condition (1) only for usual polynomials of operators instead of the matrix polynomials? That is, when the condition $k = 1, 2, \ldots, n(R)$ can be replaced by $k = 1$?

The classical Volterra operator V of integration:

$$Vf(t) = 2i \int_t^1 f(s)ds, \ f \in L^2([0,1]), \ t \in [0,1].$$

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Arveson's Question (1969) Whether the norms $\|f(V)\|_{op}$, for $f \in \mathbb{C}[t]$, determine the unitary similarity class of V in the set of irreducible compact operators on $L^2([0,1])$.

Toeplitz matrix

Theorem ([FGS'11] and [FFGSS'11])

Let $R \in M_n(\mathbb{C})$ be an upper triangular Toeplitz matrix

$$\begin{bmatrix} a_0 & a_1 & a_2 & \dots & a_{n-1} \\ & a_0 & a_1 & \ddots & \vdots \\ & & \ddots & \ddots & a_2 \\ & & & a_0 & a_1 \\ 0 & & & & a_0 \end{bmatrix}, \quad \text{whith} \quad a_1 \neq 0,$$

and $S \in M_n(\mathbb{C})$ be any matrix. Then R and S are unitarily similar if and only if

$$\|f(R)\| = \|f(S)\| \quad \text{for all} \quad f \in \mathbb{C}[t] \quad \text{of degree at most } n-1,$$

where $\|\cdot\|$ is the operator norm or Frobenius norm

Normal matrix

Theorem [G'12]

Let $R \in M_n(\mathbb{C})$ be a normal matrix and $S \in M_n(\mathbb{C})$ be any matrix. Then R and S are unitarily similar if and only if

$$\|f(R)\|_F = \|f(S)\|_F \quad \text{for all} \quad f \in \mathbb{C}[t] \quad \text{of degree at most } n-1,$$

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In case of the operator norm, this statement is not true!

Counterexample 1

Consider the following normal operators acting on $\mathbb{C}^3$:

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad \text{and} \quad S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Then the equality of operator norms holds, i.e.

$$\|f(R)\|_{op} = \|f(S)\|_{op} \quad \text{for all } f \in \mathbb{C}[t],$$

but R and S are not unitarily similar.

Counterexample 2

Consider the following irreducible nilpotent operators acting on $\mathbb{C}^3$:

$$R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad S = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then the equality of norms holds, i.e.

$$\|f(R)\| = \|f(S)\| \quad \text{for all } f \in \mathbb{C}[t],$$

but R and S are not unitarily similar.

But their 2×2 principal minors

$$R_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad S_2 = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$$

don't satisfy this condition.
Thus it is natural to check the equality of norms not only for $n \times n$ matrices R and S, but also for all their $k \times k$ principal minors R_k and S_k:

$$\|f(R_k)\| = \|f(S_k)\| \quad \text{for all } f \in \mathbb{C}[t], \quad k = 1, \dots, n-1.$$

Criterion for indecomposable matrices

Theorem ([FGS'11] and [FFGSS'11])

Let R and S be $n \times n$ upper triangular matrices that are not similar to direct sums of square matrices of smaller sizes. Then R and S are unitarily similar if and only if

$$\|f(R_k)\| = \|f(S_k)\| \qquad \text{for all } f \in \mathbb{C}[x] \text{ and } k = 1, \ldots, n,$$

where R_k and S_k are $k \times k$ principal submatrices of R and S, and $\| \cdot \|$ is the operator norm or Frobenius norm.

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where R_k and S_k are $k \times k$ principal submatrices of R and S, and $\|\cdot\|$ is the operator norm or Frobenius norm.

We consider only upper triangular matrices because of the Schur unitary triangularization theorem: *every square matrix A is unitarily similar to an upper triangular matrix B whose diagonal entries are complex numbers in any prescribed order*, say, in the lexicographical order:

$$a + bi \preccurlyeq c + di \quad \text{if either } a < b, \text{ or } a = b \text{ and } b \leqslant d.$$

Counterexample 3

The theorem cannot be extended to matrices with several eigenvalues in the case of Frobenius norm. Consider the following two operators acting on $\mathbb{C}^4$:

$$R = \begin{bmatrix} 0 & 1 & -1 & a \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix} \quad \text{and} \quad S = \begin{bmatrix} 0 & 1 & -1 & b \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix},$$

where $a \neq b, |a| = |b| = 1$.
Then the equality of norms holds, i.e.

$$\|f(R_k)\|_F = \|f(S_k)\|_F \qquad \text{for all } f \in \mathbb{C}[x] \text{ and } k = 1, \dots, n,$$

where R_k and S_k are $k \times k$ principal submatrices of R and S, but R and S are not unitarily similar.

Matrices in general position

Let

$$X_n := \begin{bmatrix} x_{11} & \dots & x_{1n} \\ & \ddots & \vdots \\ 0 & & x_{nn} \end{bmatrix}$$

be a matrix whose upper triangular entries are variables.
Denote by $\mathbb{C}[x_{ij} | i \leqslant j \leqslant n]$ the set of polynomials in these variables.
For simplicity of notation, we write $f\{X_n\}$ instead of $f(x_{11}, x_{12}, x_{22}, \dots)$.
For each $f \in \mathbb{C}[x_{ij} | i \leqslant j \leqslant n]$, write

$$M_n(f) := \{A \in \mathbb{C}^{n \times n} \,|\, A \text{ is upper triangular and } f\{A\} \neq 0\}.$$

For example, if

$$f\{X_n\} := x_{12}x_{23}\cdots x_{n-1,n}\prod_{i<j}(x_{ii}-x_{jj}),$$

then $M_n(f)$ consists of matrices of the form

$$\begin{bmatrix} \lambda_1 & a_{12} & \dots & a_{1n} \\ & \lambda_2 & \ddots & \vdots \\ & & \ddots & a_{n-1,n} \\ 0 & & & \lambda_n \end{bmatrix}, \qquad \begin{matrix} \lambda_i \neq \lambda_j \text{ if } i \neq j, \\ \text{all } a_{i,i+1} \neq 0. \end{matrix}$$

We say that $n \times n$ upper triangular matrices *in general position* possess some property if there exists a nonzero polynomial $f \in \mathbb{C}[x_{ij} | i \leqslant j \leqslant n]$ such that all matrices in $M_n(f)$ possess this property. Thus, this property holds for all matrices in $\mathbb{C}^{n\times n}$ except for matrices from an algebraic variety of smaller dimension.

Criterion for matrices in general position

Theorem [FFGSS'11]

Two $n \times n$ upper triangular matrices R and S in general position with lexicographically ordered eigenvalues on the main diagonal are unitarily similar if and only if

$$\|f(R_k)\|_F = \|f(S_k)\|_F \qquad \text{for all } f \in \mathbb{C}[x] \text{ and } k = 1, \ldots, n,$$

where R_k and S_k are $k \times k$ principal submatrices.

Criterion for unicellular operators

Theorem [FGS'11]

Let R and S be compact operators on a complex separable Hilbert space such that all their invariant subspaces form chains, i.e.

$$\begin{matrix} 0 \subset U_1 \subset U_2 \subset \dots \\ 0 \subset V_1 \subset V_2 \subset \dots \end{matrix} \qquad \dim U_i = \dim V_i = i.$$

Then the operators R and S are unitarily similar if and only if

$$\|f(R\,|_{U_i})\|_{op} = \|f(S\,|_{V_i})\|_{op} \quad \text{for all } f \in \mathbb{C}[t] \quad \text{and} \quad i = 1, 2, \dots.$$

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An operator $R \in B(H)$ is called unicellular if for every two invariant closed subspaces $\mathfrak{L}$ and $\mathfrak{M}$ of the operator R we have:

$$\mathfrak{L} \subset \mathfrak{M} \quad \text{or} \quad \mathfrak{M} \subset \mathfrak{L}.$$

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In case of a finite-dimensional Hilbert space an operator is unicellular if its matrix is similar to a Jordan block.

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If the Hilbert space is finite-dimensional then the criterion holds for all unicellular operators.

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Theorem [FGS'11]

For every $\varepsilon > 0$ there is a finite-dimensional subspace $L \subset L^2([0,1])$ such that if P denotes the projection onto L, then

1. $PVP_{|L}$ is a unicellular operator whose unitary similarity orbit, as an operator on L, is completely determined by the family of norms $\{\|f(PVP_{|L})\| : f \in \mathbb{C}[t]\}$
2. $\|PVP - V\| < \varepsilon$.

Thanks for your attention!