3D gravity with propagating torsion: the AdS sector

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Outline

The talk is based on the paper:

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- In our attempts to properly understand basic aspects of the gravitational dynamics at both classical and quantum level, we are naturally led to consider three-dimensional (3D) gravity as a technically simpler model with the same conceptual features.
- Following a traditional approach based on general relativity (GR), 3D gravity has been studied mainly in the realm of Riemannian geometry.
- In the early 1990s, a new approach to 3D gravity, based on a modern gauge-field theoretic conception of gravity characterized by a Riemann-Cartan geometry of spacetime (in which both the *torsion* and the *curvature* carry the dynamics of gravity), was initiated by Mielke and Baekler (MB).

- The Mielke-Baekler (MB) model is introduced as a topological 3D gravity with torsion, with an idea to explore the influence of geometry on the dynamics of gravity.
- Recent investigations along these lines led to remarkable results:
  - the MB model possesses the black hole solution,
  - it can be formulated as a Chern-Simons gauge theory,
  - in the AdS sector, asymptotic symmetry is described by two independent Virasoro algebras with different central charges,
  - the black hole entropy is found to depend on torsion, and
  - the geometric idea of torsion is compatible with supersymmetry

- Einstein's GR in 3D, with or without a cosmological constant, is also a topological theory, which has no propagating degrees of freedom.
- Such a degenerate situation is not quite a realistic feature of the gravitational dynamics.
- Thus, one is naturally motivated to study gravitational models *with propagating degrees of freedom*. In the context of Riemannian geometry, there are two well-known models of this type: topologically massive gravity (TMG), and the Bergshoeeff–Hohm–Townsend (BHT) gravity.
- In 3D gravity with torsion, an extension that includes propagating modes is even more natural—it corresponds to Lagrangians which are *quadratic* in the field strengths, as in the standard gauge approach.

- The Lagrangian of the theory has a rather large number of parameters, and one is faced with the problem of choice of a set of parameters which defines an acceptable gravitational model.
- Motivated by the actual importance of massive gravity in high-energy physics and cosmology, Hernaski et al. used the spin projection operators to investigate how the existence of propagating torsion can be used to build up a unitary massive gravity model of the BHT type.
- After that, Helayël-Neto et al. studied the parity-preserving 3D gravity with propagating torsion combined with the Chern-Simons term; using the requirements of no ghosts and no tachions, they found certain restrictions on the parameters.

- Although these arguments are commonly accepted in the literature, one should note that they essentially rely on the weak-field approximation of the theory, in which inherently nonlinear properties of gravity remain untouchable.
- Our approach to 3D gravity with propagating torsion is aimed at studying essential aspects of its nonlinear dynamics.
- We start by introducing basic elements of the Lagrangian formalism, whereupon we focus our attention on the AdS sector, examining the existence of black holes and the nature of asymptotic symmetries.
- In future investigations we shall use the criterion of stability of the canonical structure under linearization to find out the restrictions on parameters that define viable PGT models.

- Our conventions are as follows:
  - the Latin indices $(i, j, k, ...)$ refer to the local Lorentz frame, the Greek indices $(\mu, \nu, \lambda, ...)$ refer to the coordinate frame, and both run over 0,1,2;
  - the metric components in the local Lorentz frame are $\eta_{ij} = (+, -, -)$; totally antisymmetric tensor ε^{ijk} is normalized to $\varepsilon^{012} = 1$.
- Our notation follows the PGT framework:
  - fundamental dynamical variables are the triad field b^i and the Lorentz connection $A^{ij} = -A^{ji}$ (1-forms),
  - $T^i = db^i + \varepsilon^{ijk}\omega_j b_k$ and $R^{ij} = dA^{ij} + A^i{}_k A^{kj}$ are the torsion and the curvature (2-forms),
  - The antisymmetry of the Lorentz connection A^{ij} implies that the geometric structure of PGT corresponds to a Riemann-Cartan geometry.

- General dynamics of 3D gravity with propagating torsion is defined by the Lagrangian 3-form

$$L = L_G(b^i, T^i, R^{ij}) + L_M(b^i, \psi, \nabla\psi) \tag{2.1a}$$

where L_M is matter contribution, and gravitational piece L_G is at most quadratic in torsion and curvature. Assuming that L_G preserves parity, we have

$$\begin{aligned} L_G &= -a\varepsilon_{ijk} b^i \wedge R^{jk} - \frac{1}{3}\Lambda_0 \varepsilon_{ijk} b^i \wedge b^j \wedge b^k + L_{T^2} + L_{R^2} \,, \\ L_{T^2} &= T^i \wedge {}^\star \left(a_1 {}^{(1)}T_i + a_2 {}^{(2)}T_i + a_3 {}^{(3)}T_i \right) \,, \\ L_{R^2} &= \frac{1}{2} R^{ij} \wedge {}^\star \left(b_4 {}^{(4)}R_{ij} + b_5 {}^{(5)}R_{ij} + b_6 {}^{(6)}R_{ij} \right) \,, \end{aligned} \tag{2.1b}$$

where ${}^{(a)}T_i$ and ${}^{(a)}R_{ij}$ are irreducible components.

- Let us now introduce the covariant gravitational momenta:

$$
\begin{aligned}
H_i &:= \frac{\partial L_G}{\partial T^i} = 2^\star \left(a_1{}^{(1)}T_i + a_2{}^{(2)}T_i + a_3{}^{(3)}T_i \right) , \\
H_{ij} &:= \frac{\partial L_G}{\partial R^{ij}} = -2a\varepsilon_{ijk}b^k + H'_{ij} , \\
H'_{ij} &= 2^\star \left(b_4{}^{(4)}R_{ij} + b_5{}^{(5)}R_{ij} + b_6{}^{(6)}R_{ij} \right) .
\end{aligned} \tag{2.2}
$$

- We also define the dynamical energy-momentum and spin currents (2-forms) for the gravitational field:

$$
t_i := \frac{\partial L_G}{\partial b^i} , \qquad s_{ij} := \frac{\partial L_G}{\partial A^{ij}} , \tag{2.3}
$$

as well as the corresponding matter currents (2-forms):

$$
\tau_i := \frac{\partial L_M}{\partial b^i} , \qquad \sigma_{ij} := \frac{\partial L_M}{\partial A^{ij}} = \Sigma_{ij}\psi \frac{\partial L_M}{\partial \nabla \psi} . \tag{2.4}
$$

- The variation of the Lagrangian (2.1a) with respect to b^i and A^{ij} produces the following gravitational field equations:

$$\nabla H_i + t_i = -\tau_i\,, \tag{2.5a}$$

$$\nabla H_{ij} + s_{ij} = -\sigma_{ij}\,. \tag{2.5b}$$

- The second field equation can be rewritten in an equivalent form as:

$$-2a\varepsilon_{ijk}T^k + \nabla H'_{ij} + s_{ij} = -\sigma_{ij}\,. \tag{2.5b'}$$

- Using the expressions for the gravitational field momenta, the gravitational Lagrangian can be rewritten in a more compact form as:

$$L = \frac{1}{2}T^i H_i + \frac{1}{2}R^{ij}(-2a\varepsilon_{ijk}b^k) + \frac{1}{4}R^{ij}H'_{ij} - \frac{1}{3}\Lambda_0\varepsilon_{ijk}b^i b^j b^k\,. \tag{2.6}$$

- The weak-field approximation of 3D gravity with propagating torsion around the Minkowski background M_3 yields an approximate picture of the gauge structure and dynamical content of the theory.
- The particle spectrum of 3D gravity with propagating torsion can be studied either by using an extended basis of the spin projection operators (Helayël-Neto et. al.) or an approach based on the covariant field equations.
- The propagating modes of the theory are those associated to the 9 independent modes of the Lorentz connection $A^{ij}{}_{\mu}$. By subtracting 3 gauge degrees of freedom, corresponding to 3 local Lorentz rotations, one finds that at most 6 degrees of freedom can be physical.

- For massive modes, the spin content can be determined by looking at the corresponding (one-dimensional) irreducible representations of the little (Abelian) group $SO(2)$.
- In two spatial dimensions parity is defined as the inversion of one axis.
- Since parity is a symmetry of the Lagrangian , the structure of the corresponding irreducible representations is changed: they contain two states with the same value of spin J, which transform into each other under P.
- The irreducible representations of $SO(2) \times P$ provide a foundation for understanding the particle content of any Lorentz-covariant field theory in 3D. It is usually simpler to start with finite-dimensional representations of the full Lorentz group $SO(1,2)$.

- The weak-field approximation around M_3 takes the form

$$b^i{}_\mu = \delta^i_\mu + \tilde{b}^i{}_\mu \,, \qquad A^{ij}{}_\mu = \tilde{A}^{ij}{}_\mu \,, \tag{3.1a}$$

where the tilde sign denotes small field excitations. Then:

$$\begin{aligned} T^i{}_{\mu\nu} &= \tilde{T}^i{}_{\mu\nu} + \mathcal{O}_2 \,, \qquad \tilde{T}^i{}_{\mu\nu} = \partial_\mu \tilde{b}^i{}_\nu - \partial_\nu \tilde{b}^i{}_\mu + 2\tilde{A}^i{}_{[\nu\mu]} \,, \\ R^{ij}{}_{\mu\nu} &= \tilde{R}^{ij}{}_{\mu\nu} + \mathcal{O}_2 \,, \qquad \tilde{R}^{ij}{}_{\mu\nu} = \partial_\mu \tilde{A}^{ij}{}_\nu - \partial_\nu \tilde{A}^{ij}{}_\mu \,. \end{aligned} \tag{3.1b}$$

The linearized field equations in vacuum take the form:

$$\partial_\mu \tilde{H}_i{}^{\mu\nu} - 2a\tilde{G}^\nu{}_i = 0 \,, \tag{3.2a}$$

$$-a\varepsilon^{\mu\nu\rho}_{ijk} \tilde{T}^k{}_{\mu\rho} + \partial_\mu \tilde{H}^{(2)\mu\nu}_{ij} + 2\tilde{H}_{[ij]}{}^\nu = 0 \,. \tag{3.2b}$$

- The spectrum of excitations around M_3 consists of *6 independent torsion modes*: two spin-$0^{\mp}$ states (a,σ), two spin-1 states $\bar{v}_i$, and two spin-2 states χ_{ij}, defined by:

$$
\begin{aligned}
&a = \frac{1}{6}\varepsilon_{ijk}T^{ijk}\,, \quad m^2_{0^-} = \frac{3(a-a_1)(a+2a_3)}{(a_1+2a_3)b_5}\,,\\
&\sigma = \partial^i v_i \equiv \partial^i T_{ij}{}^j\,, \quad m^2_{0^+} = \frac{3a(a+a_2)}{a_2(b_4+2b_6)}\,,\\
&\bar{v}^i := v^i + \frac{1}{m^2_{0^+}}\partial^i\sigma\,, \quad m^2_1 = \frac{4(a-a_1)(a+a_2)}{(a_1+a_2)(b_4+b_5)}\,,\\
&\chi_{ij} = \partial^k t_{k(ij)} + \frac{a_1(a+a_2)^2}{2(a-a_1)\left[a(a_2-a_1)-2a_1a_2\right]}\partial_{(i}\bar{v}_{j)}\,,\\
&m^2_2 = -\frac{a(a-a_1)}{a_1b_4}\,.
\end{aligned}
\tag{3.3}
$$

- In the MB model of 3D gravity with torsion, there exists an interesting vacuum solution, the black hole with torsion, defined by the pair $(b^i, \omega^i := \frac{1}{2}\varepsilon^{ijk}A_{jk})$:

$$
\begin{aligned}
&b^0 = Ndt\,,\ b^1 = N^{-1}dr\,,\ b^2 = r(d\varphi + N_\varphi dt) \qquad (4.1a)\\
&N^2 = -8mG + \frac{r^2}{\ell^2} + \frac{16G^2J^2}{r^2}\,, \quad N_\varphi = \frac{4GJ}{r^2}\,,\\
&\omega^i = \tilde{\omega}^i + \frac{p}{2}b^i\,. \qquad (4.1b)
\end{aligned}
$$

where $\tilde{\omega}^i$ is the Riemannian connection.

- The field strengths have the following form:

$$2T_i = p\varepsilon_{ijk}b^j b^k\,, \qquad 2R_i = q\varepsilon_{ijk}b^j b^k\,, \qquad (4.2)$$

where p and q are parameters, while the effective cosmological constant is $\Lambda_{\text{eff}} = q - \frac{1}{4}p^2 = -\frac{1}{\ell^2} < 0$.

- By combining (4.2) with the field equations in vacuum, we can obtain the restrictions on p and q, under which the BTZ-like black hole (or AdS_3) is an exact solution:

$$aq-\Lambda_0+\frac{1}{2}p^2a_3-\frac{1}{2}q^2b_6=0\,,\quad p(a+qb_6+2a_3)=0\,.$$

- The second equation leads to the following two cases:

  a) $p=0 \quad \Rightarrow \quad$ for $b_6\neq 0$, we have

$$qb_6=a\pm\sqrt{a^2-2b_6\Lambda_0}\,.$$

  For $b_6=0$, the value of q is $q=\Lambda_0/a$.

  b) $a+b_6q+2a_3=0 \quad \Rightarrow$

$$\frac{1}{2}a_3p^2=\Lambda_0+\frac{1}{2}q(qb_6-2a)=\Lambda_0+\frac{1}{2b_6}(2a_3+a)(2a_3+3a)\,.$$

  For $a_3=0$, p remains undetermined, which is physically not acceptable.

- Following Nester's ideas, based on the analogy with the first order formulation of electrodynamics, we introduce the *first order formulation* of our theory by:

$$L = T^i\tau_i + R^i\rho_i - V(b^i,\tau_i,\rho'_i) - \frac{1}{3}\Lambda_0\varepsilon_{ijk}b^ib^jb^k + L_M\,, \quad (5.1)$$

  where τ_i and $\rho_i := 2ab_i + \rho'_i$ are the covariant field momenta (1-forms), conjugate to b^i and ω^i.
- τ_i and ρ_i are also *independent* dynamical variables.
- The potential V is quadratic in τ and ρ', and its form is chosen so as to ensure the on-shell equivalence of the new formulation (5.1) with our theory.
- The first order formulation leads to a particularly simple construction of the gauge generator.

- The canonical gauge generator takes the form:

$$
\begin{aligned}
G &= -G_1 - G_2\,, \\
G_1 &= \dot{\xi}^\mu \left(b^i{}_\mu \pi_i{}^0 + \omega^i{}_\mu \Pi_i{}^0 + \tau^i{}_\mu p_i{}^0 + \rho^i{}_\mu P_i{}^0 \right) \\
&+ \xi^\mu \Big[b^i{}_\mu \hat{\mathcal{H}}_i + \omega^i{}_\mu \mathcal{K}_i + \tau^i{}_\mu \hat{\mathcal{T}}_i + \rho^i{}_\mu \hat{\mathcal{R}}_i \\
&+ (\partial_\mu b^i{}_0)\pi_i{}^0 + (\partial_\mu \omega^i{}_0)\Pi_i{}^0 + (\partial_\mu \tau^i{}_0) p_i{}^0 + (\partial_\mu \rho^i{}_0) P_i{}^0 \Big]\,, \\
G_2 &= \dot{\theta}^i \Pi_i{}^0 + \theta^i \left[\mathcal{K}_i - \varepsilon_{ijk} \left(b^j{}_0 \pi^{k0} + \omega^j{}_0 \Pi + \tau^j{}_0 p^{k0} + \rho^j{}_0 P^{k0} \right) \right]\,,
\end{aligned}
$$

where $(\pi_i{}^0, \Pi_i{}^0, p_i{}^0, P_i{}^0)$ are canonical momenta (primary constraints) corresponding to $(b^i{}_0, \omega^i{}_0, \tau^i{}_0, \rho^i{}_0)$ and $(\hat{\mathcal{H}}_i, \mathcal{K}_i, \hat{\mathcal{T}}_i, \hat{\mathcal{R}}_i)$ are corresponding secondary constraints.

- When the canonical structure of the theory is completely known, the canonical gauge generator G can be constructed using the well-known Castellani's algorithm.
- However, it is sometimes simpler to use an alternative method, based on the following criterion:
  - a phase-space functional G is the canonical gauge generator if it produces the correct gauge transformations of all the phase-space variables.
- The proof of this conjecture is presented in our paper. Indeed, looking at the action of G on the phase-space variables φ in R, defined by $\delta_0^*\varphi = \{\varphi, G\}^*$, one finds that these gauge transformations coincide with the Poincaré gauge transformations on shell.

- AdS asymptotic behavior is defined by the requirements:
  (a) asymptotic configurations should include the black hole geometries;
  (b) they should be invariant under the action of the AdS group;
  (c) asymptotic symmetries should have well-defined generators.
- The requirements (a) and (b) lead to the following asymptotic form of the triad field and the connection :

$$b^i{}_\mu = \begin{pmatrix} \dfrac{r}{\ell} + \mathcal{O}_1 & O_4 & O_1 \\ \mathcal{O}_2 & \dfrac{\ell}{r} + \mathcal{O}_3 & O_2 \\ \mathcal{O}_1 & \mathcal{O}_4 & r + \mathcal{O}_1 \end{pmatrix}, \quad \omega^i{}_\mu \sim \tilde{\omega}^i{}_\mu + \frac{p}{2}\tilde{b}^i{}_\mu \,.$$

- The AdS asymptotic behavior of the covariant momenta is:

$$\tau^i{}_\mu \sim 2pa_3\tilde{b}^i{}_\mu \,, \qquad \rho^i{}_\mu \sim 2(a + qb_6)\tilde{b}^i{}_\mu \,.$$

- A closer inspection of these formulas shows a remarkable similarity to the corresponding structure in the MB model with *vanishing* Chern–Simons coupling constant ($\alpha_3 = 0$), which we denote by MB′.
- In MB′ we have $\tau_i = \alpha_4 b^i$ and $\rho_i = 2ab^i$.
- Thus, although MB′ and 3D gravity with propagating torsion have rather different structures in the bulk, their AdS asymptotic behaviors are related by a simple correspondence:

$$\alpha_4 \leftrightarrow 2pa_3\,, \qquad a \leftrightarrow a + qb_6\,. \tag{6.1}$$

Clearly, these relations shed a new light on the AdS asymptotic structure of 3D gravity with propagating torsion.

- The canonical generator, which acts on dynamical variables via the DB operation, has to be a differentiable phase-space functional. For a given set of asymptotic conditions, this property is ensured by adding suitable surface terms to G:

$$\tilde{G} = G + \Gamma\,, \quad \Gamma = -\int_0^{2\pi} d\varphi \left(\xi^0\mathcal{E}^1 + \xi^2\mathcal{M}^1\right)\,, \tag{6.2a}$$

- The values of the improved generators of time translations and spatial rotations are given by the corresponding surface terms, which define the conserved charges of the system—the energy and the angular momentum:

$$E = \int_0^{2\pi} d\varphi\mathcal{E}^1\,, \qquad M = \int_0^{2\pi} d\varphi\mathcal{M}^1\,. \tag{6.2b}$$

- In particular, the energy and angular momentum of the BTZ-like black hole are:

$$E = \left(1 + \frac{qb_6}{a}\right) m \,, \qquad M = \left(1 + \frac{qb_6}{a}\right) J \,. \tag{6.3}$$

- We have checked that Nester's covariant approach, yields the same result.
- The conserved charges depend on the curvature strength q, but not on the torsion strength p. For $q \neq 0$, the effect of the quadratic curvature terms in the Lagrangian is to rescale the values of the black hole charges as compared to the GR expressions.

- The asymptotic canonical algebra of the improved generators $\tilde{G}$ takes the form of two independent Virasoro algebras:

$$i\{L_n^{\mp}, L_m^{\mp}\} = (n-m)L_{n+m}^{\mp} + \frac{c^{\mp}}{12} n^3 \delta_{n,-m}\,, \tag{6.4a}$$

where $c^{\mp}$ are classical central charges:

$$c^- = c^+ = \left(1 + \frac{qb_6}{a}\right)\frac{3\ell}{2G}\,. \tag{6.4b}$$

- Once we have the central charges, we can use Cardy's formula to calculate the black hole entropy:

$$S = \left(1 + \frac{qb_6}{a}\right)\frac{2\pi r_+}{4G}\,, \tag{6.5}$$

where r_+ is the radius of the outer black hole horizon.

- We studied dynamical structure of 3D gravity with propagating torsion.
- Starting with the generic parity-preserving PGT Lagrangian, we derived the general form of the field equations and analyzed its particle content.
- Focusing on the fully nonlinear aspects in the AdS sector, we found the conditions on parameters that ensure to have the BTZ-like black hole as an exact solution of the model.
- By examining the AdS asymptotic structure, we obtained the expressions for the conserved charges of the black hole with torsion, whereas the asymptotic symmetry is found to be described by two independent Virasoro algebras with equal central charges.