New symmetries and particular solutions for the 2D Black-Scholes equation

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Abstract: *Starting from a general second order differential equation, the Lie symmetry analysis of the* $2D$ *Black-Scholes model for option pricing is performed. The corresponding Lie algebra is identified and the algorithm for constructing exact (invariant) solutions under one-dimensional and two-dimensional subalgebras of this Lie algebra is illustrated. By applying the inverse symmetry problem, more general* $2D$ *equivalent models, as symmetry group, with Black-Scholes or Jacobs-Jones ones, are pointed out.*
Keywords*: option pricing models, symmetry analysis*

Preliminary remarks

- Description of social and of individual wealth – topic of interest not only for economists, but for mathematicians and physicists, too.

- Several techniques of fundamental physics, coming from quantum mechanics, nonlinear dynamics, field theory and other related tools, have been applied in finance and in all areas of economics. Common golden thread: stochastic processes.

- The *option pricing theory*, one of the most successful theories which describe various phenomena of financial markets as evolution of value for financial derivatives, stock price patterns, critical crashes etc., is essentially based on equations coming from Physics.

  Example 1: *Black-Scholes formula* has been derived starting from the heat equation and using Ito's lemma for Brownian motion.
  (*Fischer Black and Myron Scholes, Nobel Prize for Economics in 1997, together with Robert Merton*)

  Example 2: *Dragulescu and Yakovenko models* of money, wealth and income distributions start from the claim that the probability distribution follows the Boltzmann-Gibbs law.

I.1.The concept of integrability for dynamical systems

- Dynamical systems are described by nonlinear differential equations. If solutions exist, the diffential system is said to be integrable.

- Sometime it is difficult to find a complete set of solutions and it is quite enough if one can decide on the integrability of the system.

  Methods: the Hirota's bilinear method, the Backlund transformation method, the inverse scattering method, the Lax pair operator, the Painleve analysis, etc.

- To decide if a nonlinear differential equation is integrable, one of the following situation should appear:

(i) the existence of a number of functionally independent first integrals/invariants equal to the order of the system in general and half that for a Lagrangian system as a consequence of Liouville's Theorem;
(ii) the existence of a sufficient number of Lie symmetries to reduce the partial differential equation to an ordinary differential equation;
(iii) Painlevé test.

I.2.The symmetry method for solving dynamical systems

- There is no a general theory for completely solving of the nonlinear pdes.
- One of the most useful techniques - *the symmetry method*. It is useful for:
  - symmetry reduction of differential equations and thus obtain classes of exact solutions.
  - as symmetry transforms solutions into solutions, one can generate new solutions from known ones.
- Initially: the *Lie (classical) symmetry method (CSM).*

For a n -th order partial differential system:

$$\Delta_\nu(x, u^{(n)}[x]) = 0$$

$x \equiv \{x^i, \quad i = \overline{1,p}\} \subset R^p$ independent variables, $u \equiv \{u^\alpha, \alpha = \overline{1,q}\} \subset R^q$ the dependent ones. The notation $u^{(n)}$ designates the set of variables which includes u and the partial derivatives of u up to n -th order.

The general infinitesimal symmetry operator has the form:

$$X = \sum_{i=1}^{p} \xi^i(x,u)\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \phi_\alpha(x,u)\frac{\partial}{\partial u^\alpha}$$

The n -th extension is:

$$X^{(n)} = X + \sum_{\alpha=1}^{q}\sum_{J} \phi_\alpha^J(x,u^{(n)})\frac{\partial}{\partial u_J^\alpha}$$

Notations:

$$u_J^\alpha = \frac{\partial^m u^\alpha}{\partial x^{j_1}\partial x^{j_2}..\partial x^{j_m}}$$

$$\phi_\alpha^J(x^i, u^{(n)}) = \mathcal{D}_J[\phi_\alpha - \sum_{i=1}^{p} \xi^i u_i^\alpha] + \sum_{i=1}^{p} \xi^i u_{J,i}^\alpha,\ \alpha = \overline{1,q}$$

$$u_i^\alpha = \frac{\partial u^\alpha}{\partial x^i},\ i = \overline{1,p}$$

$$u_{J,i}^\alpha = \frac{\partial u_J^\alpha}{\partial x^i} = \frac{\partial^{m+1} u^\alpha}{\partial x^i \partial x^{j_1} \partial x^{j_2} .. \partial x^{j_m}}$$

$$\mathcal{D}_J = \mathcal{D}_{j_1}\mathcal{D}_{j_2}\ldots\mathcal{D}_{j_m} = \frac{d^{\ m}}{dx^{j_1} dx^{j_2} .. dx^{j_m}}$$

- The invariance condition is:

$$X^{(n)}[\Delta]?_{\Delta=0} = 0$$

- The characteristic equations associated to the general symmetry generator have the form:

$$\frac{dx^1}{\xi^1} = \ldots = \frac{dx^p}{\xi^p} = \frac{du^1}{\phi_1} = \ldots = \frac{du^q}{\phi_q}$$

- By integrating one get the invariants $I_r,\quad r = \overline{1,(p+q-1)}$ of the analyzed system.
- Similarity reduction: the set of similarity variables is found in terms of which the original evolutionary equation with p independent variables and q dependent ones can be reduced to a set of differential equations with $(p+q-1)$ variables.

Generalizations of the Lie symmetry method:

1. The *non-classical symmetry method* (Bluman and Cole): added the invariance surface condition:

$$Q^{\alpha}(x, u^{(1)}) \equiv \phi_{\alpha}(x,u) - \sum_{i=1}^{p} \xi^{i}(x,u)\frac{\partial u^{\alpha}}{\partial x^{i}} = 0,\ \alpha = \overline{1,q}$$

   Consequences:
   - Smaller number of determining equations for the infinitesimals $\xi^{i}(x,u), \quad \phi_{\alpha}(x,u),$
   - More solutions than the CSM (any classical symmetry is a non-classical one)
2. The *direct method* (Clarkson and Kruskal): a direct, algorithmic method for finding symmetry reductions.
3. The *differential constraint approach* (Olver and Rosenau): the original system of partial differential equations can be enlarged by appending additional differential constraints (side conditions), resulting an over-determined system of partial differential equations.
4. The *generalized conditional symmetries method* or *conditional Lie-Bäcklund symmetries* (Fokas, Liu and Zhdanov).

I.4.The inverse symmetry problem

- *The **direct symmetry problem*** of evolutionary equations consists in:
- Determining the Lie symmetry group corresponding to a given evolutionary equation.
- Obtaining the invariants associated to each symmetry operator (with the characteristic equations).
- Obtaining some reduced equations with the similarity reduction procedure.
- Solving the reduced equation and generating the similarity solution of the analyzed model.

- The ***inverse symmetry problem***: what is the largest class of evolutionary equations which are equivalent from the point of view of their symmetries?
- So, this problem supposes to impose a concrete symmetry group to a general analyzed model. With this condition, the general symmetry determining equations could be solved and allow to determine all concrete models which admit the same Lie symmetry group.
- The example of a $2D$ dynamical system:

$$u_t = A(x,y,t,u)u_{xy} + B(x,y,t,u)u_x u_y + C(x,y,t,u)u_{2x} + D(x,y,t,u)u_{2y} + +E(x,y,t,u)u_y + F(x,y,t,u)u_x + G(x,y,t,u) \qquad \text{(I.1)}$$

with $A(x,y,t,u),\ B(x,y,t,u),\ C(x,y,t,u),\ D(x,y,t,u),\ E(x,y,t,u),\ F(x,y,t,u),\ G(x,y,t,u)$ arbitrary functions.

- The general expression of the Lie symmetry operator with $\varphi \equiv 1.$:

$$X(x,y,t,u) = \frac{\partial}{\partial t} + \xi(x,y,t,u)\frac{\partial}{\partial x} + \eta(x,y,t,u)\frac{\partial}{\partial y} + \phi(x,y,t,u)\frac{\partial}{\partial u}$$

- The symmetry invariance condition is given by the relation:

$$0 = X^{(2)}[u_t - A(x,y,t,u)u_{xy} - B(x,y,t,u)u_x u_y - C(x,y,t,u)u_{2x} - D(x,y,t,u)u_{2y} - E(x,y,t,u)u_y - F(x,y,t,u)u_x - G(x,y,t,u)]$$

- The previous relation has the equivalent expression:

$$0 = -A_t u_{xy} - B_t u_x u_y - C_t u_{2x} - D_t u_{2y} - E_t u_y - F_t u_x - G_t - A_x \xi u_{xy} - B_x \xi u_x u_y -$$
$$- C_x \xi u_{2x} - D_x \xi u_{2y} - E_x \xi u_y - F_x \xi u_x - G_x \xi - A_y \eta u_{xy} - B_y \eta u_x u_y - C_y \eta u_{2x} - D_y \eta u_{2y} -$$
$$- E_y \eta u_y - F_y \eta u_x - G_y \eta - A_u \phi u_{xy} - B_u \phi u_x u_y - C_u \phi u_{2x} - D_u \phi u_{2y} - E_u \phi u_y - F_u \phi u_x -$$
$$- G_u \phi + \phi^t - A\phi^{xy} - C\phi^{2x} - D\phi^{2y} - B\phi^x u_y - F\phi^x - B\phi^y u_x - E\phi^y$$

The functions $\phi^t, \phi^x, \phi^y, \phi^{2x}, \phi^{2y}, \phi^{xy}$ can be determined using well-known formulas.

- Equating with zero the coefficient functions of various monomials in derivatives of u, the following partial differential system with 11 equations is obtained:

$$0 = \xi_u;\ 0 = \eta_u;\ 0 = B\eta_x - D\phi_{2u};\ 0 = B\xi_y - C\phi_{2u}$$
$$0 = A\eta_y - \eta A_y - A_u \phi + A\xi_x - \xi A_x + +2D\xi_y + 2C\eta_x - A_t$$
$$0 = A\eta_x + 2D\eta_y - \eta D_y - \xi D_x - D_u \phi - D_t$$
$$0 = -A\phi_{2u} + B\xi_x - B\phi_u + B\eta_y - B_t - B_x \xi - B_u \phi - B_y \eta$$
$$0 = -\eta_t + F\eta_x - B\phi_x + E\eta_y - E_t - E_x \xi - E_y \eta - E_u \phi$$
$$+ A\eta_{xy} - A\phi_{xu} + C\eta_{2x} + D\eta_{2y} - 2D\phi_{yu} \qquad \text{(I.2)}$$
$$0 = -\xi_t - B\phi_y + F\xi_x + E\xi_y - F_t - F_x \xi - F_y \eta - F_u \phi$$
$$A\xi_{xy} - A\phi_{yu} + C\xi_{2x} + D\xi_{2y} - 2C\phi_{xu}$$
$$0 = \phi_t + G\phi_u - F\phi_x - E\phi_y - G_t - G_x \xi - G_y \eta - G_u \phi$$
$$- A\phi_{xy} - C\phi_{2x} - D\phi_{2y}$$

II. Applications. Option pricing theory

II.1.The 1D Black-Scholes model

Ito's formula for Brownian motion - important mathematical ingredient

Let *x(t)* be a function of *t*, which satisfies the stochastic differential equation

$$dx = a(x,t)dt + b(x,t)dz$$

where *a(x,t)* and *b(x,t)* are deterministic functions of *x* and t, and z represents a standard Brownian motion.

Let f(x,t) be a twice continuously differentiable function of x and t. Then,

$$df(x,t) = (f_t + a(x,t)f_x + \frac{1}{2}b^2(x,t)f_{xx})dt + b(x,t)f_x dz$$

The *1D Black-Scholes models* for a European option:

$$u_t + \frac{1}{2}\sigma^2 x^2 u_{2x} + rxu_x - ru = 0 \qquad \text{(II.1)}$$

$u(x,t)$ = the value of the option with a defined pay-off,
$x \in [0,+\infty)$ = the price of the underlying asset
$t \in [0,+\infty)$ = time
σ = the volatility of the underlying asset (annual volatility of asset price)
r = a constant risk-free interest rate

Note: European option gives the right but not the obligation to buy one unit of the underlying asset at a future date (called exercise date or maturity date), at a price called exercise price.

II.2.The 2D Black-Scholes model

Let us assume now the European call options on a basket of two assets x, y with mean tendencies (or expected rates of returns) $\mu_i, i=1,2$, volatilities σ_i and correlation ρ. We assume that x, y are governed by stochastic processes of the form:

$$dx = \mu_1 x dt + \sigma_1 x dW^1$$

$$dy = \mu_2 y dt + \sigma_2 y dW^2$$

$$\rho = d(W^1, W^2)$$

The option u with pay-off $u_T(x, y)$ at maturity T will satisfy a two-dimensional Black-Scholes partial differential equation in $\mathcal{R}_+^2 \times [0, T]$:

$$u_t + \mu_1 x u_x + \mu_2 y u_y + \frac{1}{2}\sigma_1^2 x^2 u_{2x} + \frac{1}{2}\sigma_2^2 y^2 u_{2y} + \rho\sigma_1\sigma_2 xy u_{xy} - ku = 0,\ k = const. \qquad \text{(II.2)}$$

$$u(x, y, T) = u_T(x, y)$$

Remark: (II.2) = (I.1) with:

$$A(x,y,t,u) = \rho\sigma_1\sigma_2 xy; B(x,y,t,u) = 0, C(x,y,t,u) = \frac{1}{2}\sigma_1{}^2 x^2$$

$$D(x,y,t,u) = \frac{1}{2}\sigma_2{}^2 y^2 E(x,y,t,u) = \mu_1 x; F(x,y,t,u) = \mu_2 y; G(x,y,t,u) \equiv -k$$

Lie symmetries analysis:

- Lie symmetry operator:

$$X = \frac{\partial}{\partial t} + \xi(x)\frac{\partial}{\partial x} + \eta(y)\frac{\partial}{\partial y} - u[\xi_x(x) + \eta_y(y)]\frac{\partial}{\partial u}$$

- As $\{\xi, \eta\}$ = arbitrary functions, we deal with an infinite number of symmetry operators.
- The action of U can be split in various sectors, depending on the concrete form of $\{\xi, \eta\}$.

- Solving the system (I.2) we come to the following general solution:

$$\xi = \frac{c_3 x}{\rho\sigma_2}[\rho\sigma_2 \ln x - \sigma_1 \ln y] + x(c_1 t + c_2)$$

$$\eta = \frac{-c_3 y}{\rho\sigma_1}[\rho\sigma_1 \ln y - \sigma_2 \ln x] + y(c_4 t + c_5)$$

$$\phi = \omega + \beta u$$

Where:

$$\beta = \frac{1}{\rho(\sigma_1)^2(\sigma_2)^2(1-\rho^2)}$$

$$\left\{\left[-c_1\frac{\rho^2\sigma_1\sigma_2}{2} + c_3\frac{\sigma_1\sigma_2}{2}\left[(\sigma_1)^2 - 2\mu_1\right](\rho^2 - 1) + c_4\rho(\sigma_1)^2\right]\ln y + \right.$$

$$\left. + \left[c_1\rho(\sigma_2)^2 - c_3\frac{\sigma_1\sigma_2}{2}\left[(\sigma_2)^2 - 2\mu_2\right](\rho^2 - 1) - c_4\rho^2\sigma_1\sigma_2\right]\ln x + \gamma\right\}$$

The algebra of the Lie operators

The previous results suggest that we may decompose the general Lie operator in the form

$$\widetilde{X} = X + X_\omega,$$

where:

$$X = \frac{\partial}{\partial t} + \left\{\frac{c_3 x}{\rho\sigma_2}[\rho\sigma_2 \ln x - \sigma_1 \ln y] + x(c_1 t + c_2)\right\}\frac{\partial}{\partial x} + \left\{\frac{-c_3 y}{\rho\sigma_1}[\rho\sigma_1 \ln y - \sigma_2 \ln x] + y(c_4 t + c_5)\right\}\frac{\partial}{\partial y} + \beta(x,y,t)u\frac{\partial}{\partial u}$$

$$X_\omega = \omega\frac{\partial}{\partial u}$$

Accordingly, the Lie algebra possesses a natural structure that may be decomposed into a direct sum:

$$\widetilde{\Lambda} = \Lambda + \Lambda_\omega = \{X\} \oplus \{X_\omega\}$$

where Λ consists in all the operators from $\widetilde{\Lambda}$ with $\omega = 0$ and Λ_ω is the set of all the operators from $\widetilde{\Lambda}$ with $c_1 = c_2 = ... = c_6 = 0.$

More precisely,

$$\Lambda_\omega = \left\{\omega(t,x,y)\frac{\partial}{\partial u} \;\middle|\; \omega \text{ solution of (ref: 1.BS)}\right\}$$

The infinite dimensional vector space of the infinitesimal symmetries is spanned by the following operators:

$$X_1 = \frac{\partial}{\partial t}$$

$$X_2 = xt\frac{\partial}{\partial x} + \frac{1}{(\sigma_1)^2(\sigma_2)^2(1-\rho^2)}\Big\{(\sigma_2)^2 \ln x - \rho\sigma_1\sigma_2 \ln y + \frac{t}{2}\Big[(\sigma_2)^2\Big[(\sigma_1)^2 - 2\mu_1\Big] - \rho\sigma_1\sigma_2\Big[(\sigma_2)^2 - 2\mu_2\Big]\Big]\Big\}u\frac{\partial}{\partial u}$$

$$X_3 = x\frac{\partial}{\partial x}$$

$$X_4 = x\left(-\frac{\sigma_1}{\rho\sigma_2}x\ln y + x\ln x\right)\frac{\partial}{\partial x} + \left(\frac{\sigma_2}{\rho\sigma_1}y\ln x - y\ln y\right)\frac{\partial}{\partial y} + \left\{\frac{\Big[2\mu_1 - (\sigma_1)^2\Big]\ln y - \Big[2\mu_2 - (\sigma_2)^2\Big]\ln x}{2\rho\sigma_1\sigma_2}u\right\}\frac{\partial}{\partial u}$$

$$X_5 = yt\frac{\partial}{\partial y} + \frac{1}{(\sigma_1)^2(\sigma_2)^2(1-\rho^2)}\Big\{(\sigma_1)^2 \ln y - \rho\sigma_1\sigma_2 \ln x + \frac{t}{2}\Big[(\sigma_1)^2\Big[(\sigma_2)^2 - 2\mu_2\Big] - \rho\sigma_1\sigma_2\Big[(\sigma_1)^2 - 2\mu_1\Big]\Big]\Big\}u\frac{\partial}{\partial u}$$

$$X_6 = y\frac{\partial}{\partial y},\ X_7 = u\frac{\partial}{\partial u},\ X_\omega = \omega(t,x,y)\frac{\partial}{\partial u}$$

The non-vanishing commutation relations:

$$[X_1,X_2] = X_3 + (k_3+k_4)X_7,\ [X_1,X_5] = X_6 + (k_6+k_7)X_7,$$
$$[X_2,X_3] = -k_1X_7,\ [X_2,X_4] = X_2 + k_9X_5,\ [X_2,X_6] = -k_2X_7,$$
$$[X_3,X_4] = X_3 + k_9X_6 + k_{10}X_7,\ [X_3,X_5] = k_2X_7,$$
$$[X_4,X_5] = X_5 - k_8X_2,\ [X_4,X_6] = -k_8X_3 + X_6 - k_{11}X_7,$$
$$[X_5,X_6] = -k_5X_7$$

In addition, when we take into consideration the generator X_ω, we obtain the following non-vanishing relations:

$$[X_1,X_\omega] = X_{\omega_t},\ [X_2,X_\omega] = X_{xt\omega_x-\omega[k_1\ln x+k_2\ln y+t(k_3+k_4)]},[X_3,X_\omega] = X_{x\omega_x},$$

$$[X_4,X_\omega] = X_{\omega_x(k_8x\ln y+x\ln x)+\omega_x(k_9y\ln x-y\ln y)-\omega(k_{10}\ln x+k_{11}\ln y)},$$

$$[X_5,X_\omega] = X_{yt\omega_y-\omega[k_5\ln y+k_2\ln x+t(k_6+k_7)]},\ [X_6,X_\omega] = X_{y\omega_y},\ [X_7,X_\omega] = X_{-\omega}$$

II.3. Invariant solutions for the model

Invariant solutions generated by one dimensional subalgebras

Let us start by the generator:

$$X^{(1)} = \frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}$$

The functionally independent invariants of this subgroup are found by integrating the characteristic equations:

$$\frac{dt}{1} = \frac{dx}{x} = \frac{dy}{0} = \frac{du}{u}$$

and are given by the expressions:

$$I_1 = y,\ I_2 = t - \ln x,\ I_3 = \frac{u}{x}$$

The invariant solution may be expressed in the form $I_3 = \Psi^{(1)}(I_1, I_2),$ or

$$u^{(1)} = x\ \Psi^{(1)}(y,z),\ \text{where } z = t - \ln x$$

Substituting into equation we obtain, for $\Psi^{(1)}(y,z),$ the partial differential equation of second order:

$$\frac{(\sigma_1)^2}{2}\Psi^{(1)}_{2z} + \frac{(\sigma_2)^2}{2}y^2\Psi^{(1)}_{2y} - \rho\sigma_1\sigma_2 y\Psi^{(1)}_{zy} + \left[1 - \mu_1 - \frac{(\sigma_1)^2}{2}\right]\Psi^{(1)}_z +$$

$$(\mu_2 + \rho\sigma_1\sigma_2)y\Psi^{(1)}_y + (\mu_1 - k)\Psi^{(1)} = 0$$

For: $\sigma_1 = \sigma_2 = k = 1/2, \quad \mu_1 = \mu_2 = 1/10, \quad \rho = 3/5.$

$$\Psi^{(1)}(y,z) = c_1 + c_2 y^{(-1)} + c_3 e^{(-31/5)z}$$

The invariant solution for the original variable:

$$u^{(1)}(t,x,y) = x[c_1 + c_2 y^{(-1)} + c_3 e^{(-31/5)t} x^{(31/5)}]$$

Other invariant solutions:

- $u^{(2)}(t,x;\gamma) = x^{\frac{1}{4}\left(\frac{1}{4}-\frac{15\gamma}{t}+\frac{25\ln x}{2t}\right)} t^{\frac{-1}{32}\left(7+\frac{9\gamma}{8}\right)} e^{\left[\frac{t^2(1024k+11)+1800\gamma^2}{1024t}\right]}$

- $u = \Phi(x)\,;\ H = \frac{\Phi'}{\Phi},\ H(x) = \frac{1}{2(\sigma_1)^2 x}\left\{(\sigma_1)^2 - 2\mu_1 - \sqrt{8k(\sigma_1)^2 + \left[(\sigma_1)^2 - 2\mu_1\right]^2}\right\}$

$$\left\{\tanh\left[\frac{\sqrt{8k(\sigma_1)^2 + \left[(\sigma_1)^2 - 2\mu_1\right]^2}\,(d_1 - \ln x)}{2(\sigma_1)^2}\right]\right\}$$

- $u = e^t\Phi(x)$ with

$$\frac{(\sigma_1)^2}{2}x^2\Phi'' + \mu_1 x\Phi' + (1-k)\Phi = 0$$

II.4 Inverse symmetry problem and generalized models

Let us choose the infinite-dimensional Lie algebra which corresponds to the following coefficient functions of the Lie operator:

$$\varphi = c_0 = const.,\ \xi = \eta = 0,\ \phi = ku + \omega(t,x,y); k = const.$$

It generates a subalgebra with the following basic operators:

$$Y_1 = \frac{\partial}{\partial t},\ Y_2 = u\frac{\partial}{\partial u},\ Y_\omega = \omega(t,x,y)\frac{\partial}{\partial u}$$

The largest class of equations which admit (5.1) as Lie algebra is given by:

$$u_t = A_1(x)A_2(y)u_{xy} + C(x)u_{2x} + D_1(x)\left[A_2(y)\right]^2 u_{2y} + E(x,y,t,u)u_y + F(x,y,t,u)u_x + \frac{k-1}{k}W(x,y)u$$

To this class belong for example financial model suggested by Jacobs and Jones:

$$u_t = \frac{1}{2}a^2x^2u_{2x} + abcxyu_{xy} + \frac{1}{2}b^2y^2u_{2y} + \left[dx\ln\frac{y}{x} - ex^{(3/2)}\right]u_x + \left[fy\ln\frac{g}{y} - hy^{(1/2)}\right]u_y - xu$$

Conclusions

- Financial mathematical models are a challenge for researchers because of their complexity and due to the fact that they cannot be reduced to completely solvable equations. Similarly with the study of other dynamical processes, the evolution of the value of financial derivatives, stock price patterns, critical crashes etc., are strongly nonlinear and present random behaviors.

- We tried to sustain this assertion, using the symmetry method in obtaining a wide class of analytical solutions for the $2D$ Black-Scholes model, a famous option pricing model.

- In fact we investigated the generalized second order nonlinear equation (2.1), which includes many equations coming from finance, as for examples Black-Scholes and Jacobs-Jones equations.

- We applied the direct symmetry approach for computing Lie algebra and some classes of invariant solutions for the $2D$ BS model.

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