Observational Signatures for Reissner-Nordström Black Hole with Significant Charge at the Galactic Center*

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Abstract

We derive an analytical expression of a shadow size as a function of a charge in the Reissner – Nordström (RN) metric. Using the derived expression we consider shadows for negative tidal charges and charges corresponding to naked singularities $q = \mathcal{Q}^2/M^2 > 1$, where $\mathcal{Q}$ and M are black hole charge and mass, respectively. An introduction of a negative tidal charge q can describes black hole solutions in theories with extra dimensions, so following the approach we consider an opportunity to extend RN metric to negative $\mathcal{Q}^2$, while for the standard RN metric $\mathcal{Q}^2$ is always non-negative. We found that for $q > 9/8$ black hole shadows disappear. Significant tidal charges $q = -6.4$ are not consistent with observations of a minimal spot size at the Galactic Center observed in mm-band, moreover, these observations demonstrate that in comparison with the Schwarzschild black hole a Reissner – Nordström black hole with a significant charge $q \approx 1$ provides a better fit of recent observational data for the black hole at the Galactic Center.

1. Introduction

Theories with extra dimensions admit astrophysical objects (supermassive black holes in particular) which are rather different from standard ones. Tests have been proposed when it would be possible to discover signatures of extra dimensions in supermassive black holes since the gravitational field may be different from the standard one in the GR approach. So, gravitational lensing features are different for alternative gravity theories with extra dimensions and general relativity.

* Work supported was supported in part by the NSF (HRD-0833184) and NASA (NNX09AV07A).

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Recently, Bin-Nun [1, 2, 3] discussed an opportunity that the black hole at the Galactic Center is described by the tidal Reissner– Nordström metric which may be admitted by the Randall–Sundrum II braneworld scenario [4]. Bin-Nun suggested an opportunity of evaluating the black hole metric analyzing (retro-)lensing of bright stars around the black hole in the Galactic Center. Doeleman et al. evaluated a size of the smallest spot for the black hole at the Galactic Center with VLBI technique in mm-band [5] (see, constraints done from previous observations [6]). Theoretical studies showed that the size of the smallest spot near a black hole practically coincides with shadow size because the spot is the envelope of the shadow [7, 8, 9]. Measurements of the shadow size around the black hole may help to evaluate parameters of black hole metric [8, 9]. We derive an analytic expression for the black hole shadow size as a function of the tidal charge for the Reissner– Nordström metric. We conclude that observational data concerning shadow size measurements are not consistent with significant negative charges, in particular, the significant tidal charge $q = Q/M^2 = -6.4$ discussed in [1, 2, 3], where the author used a little bit different notations, namely $q' = q/4$, is practically ruled out with a very high probability (the tidal charge is roughly speaking is far beyond 9σ confidence level). We also show a smaller shadow sizes in respect to estimates obtained with the Schwarzschild black hole model can be explained with the Reissner – Nordström metric with a significant charge. It was found a critical q value for shadow existence, namely for $q \leq 9/8$, Reissner – Nordström black holes have shadows while for $q > 9/8$ the shadows do not exist. Interestingly, the same critical value is responsible for a qualitative different behavior of quasinormal modes for the scattering [10] and for existence of circular orbits of neutral test particles [11].

Now there are two basic observational techniques to investigate a gravitational potential at the Galactic Center, namely, a) monitoring the orbits of bright stars near the Galactic Center to reconstruct a gravitational potential [12] (see also a discussion about an opportunity to evaluate black hole dark matter parameters in [13] and an opportunity to constrain some class of alternative theory of gravity [14]); b) in mm-band with VLBI-technique measuring a size and a shape of shadows around black hole giving an alternative possibility to evaluate black hole parameters. The formation of retro-lensing images (also known as mirage, shadows or "faces" in the literature) due to the strong gravitational field effects nearby black holes has been investigated by several authors [15, 16, 8, 9].

Another option to test a gravity in the strong field approximation is analysis of relativistic line shape as it was shown in [17]. Later on, such signatures of the Fe K_α-line have been found in the active galaxy MCG-6-30-15 [18]. Results of our simulations of iron K_α line formation are given in [19] (where we used our approach [21]), see also [22] for a more recent review of the subject.

As J. A. Wheeler coined "Black holes have no hair": it means that a black hole is characterized by only three parameters ("hairs"), its mass M, angular momentum J and charge $\mathcal{Q}$ (see, e.g. [23, 24], or [25] for a more recent review). Therefore, in principle, charged black holes can be formed, although astrophysical conditions that lead to their formation may look rather problematic. Nevertheless, one could not claim that their existence is forbidden by theoretical or observational arguments. Moreover, we will show below that observations give a hint about an existence of a significant charge, but its origin is not clear at the moment.

Charged black holes are also object of intensive studies in quantum gravity, since a static, spherically symmetrical solution of Yang-Mills-Einstein equations with fairly natural requirements on asymptotic behavior of the solutions gives a Reissner-Nordström metric [26]. The Reissner - Nordström metric thus describes a spherically symmetric black hole with a color charge (and (or) a magnetic monopole). Later on, color charges have been found for rotating black holes as well [27].

2. Basic Equations

The expression for the Reissner - Nordström metric in natural units ($G = c = 1$) has the form

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{\mathcal{Q}^2}{r^2}\right)dt^2 + \left(1 - \frac{2M}{r} + \frac{\mathcal{Q}^2}{r^2}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{1}$$

Applying the Hamilton-Jacobi method to the problem of geodesics in the Reissner - Nordström metric, the motion of a test particle in the r-coordinate can be described by following equation (see, for example, [23])

$$r^4(dr/d\lambda)^2 = R(r), \tag{2}$$

where

$$\begin{aligned} R(r) &= P^2(r) - \Delta(\mu^2 r^2 + L^2), \\ P(r) &= Er^2 - e\mathcal{Q}r, \\ \Delta &= r^2 - 2Mr + \mathcal{Q}^2. \end{aligned} \tag{3}$$

Here, the constants μ, E, L and e are associated with the particle, i.e. μ is its mass, E is energy at infinity, L is its angular momentum at infinity and e is the particle's charge.

We shall consider the motion of uncharged particles ($e = 0$) below. In this case, the expression for the polynomial $R(r)$ takes the form

$$R(r) = (E^2 - \mu^2)r^4 + 2M\mu r^3 - (\mathcal{Q}^2\mu^2 + L^2)r^2 + 2ML^2 r - \mathcal{Q}^2 L^2. \tag{4}$$

Depending on the multiplicities of the roots of the polynomial $R(r)$, we can have three types of motion in the r - coordinate [28]. In particular, by defining $r_+ = 1 + \sqrt{1 - \mathcal{Q}^2}$, we have:

(1) if the polynomial $R(r)$ has no roots for $r \geq r_+$, a test particle is captured by the black hole;

(2) if $R(r)$ has roots and $(\partial R/\partial r)(r_{max}) \neq 0$ with $r_{max} > r_+$ (r_{max} is the maximal root), a particle is scattered after approaching the black hole;
(3) if $R(r)$ has a root and $R(r_{max}) = (\partial R/\partial r)(r_{max}) = 0$, the particle now takes an infinite proper time to approach the surface $r = const$.

If we are considering a photon ($\mu = 0$), its motion in the r-coordinate depends on the root multiplicity of the polynomial $\hat{R}(\hat{r})$

$$\hat{R}(\hat{r}) = R(r)/(M^4E^2) = \hat{r}^4 - \xi^2\hat{r}^2 + 2\xi^2\hat{r} - \hat{\mathcal{Q}}^2\xi^2. \tag{5}$$

where $\hat{r} = r/M, \xi = L/(Me)$ and $\hat{\mathcal{Q}} = \mathcal{Q}/M$.

One could see from Eq. (5) and Eqs. (3) as well that the black hole charge may influence substantially the photon motion at small radii ($r \approx 1$), while the charge effect is almost negligible at large radial coordinates of photon trajectories ($r >> 1$). In the last case we should keep in mind that the charge may cause only small corrections on photon motion.

3. Derivation of shadow size as a function of charge

Let us consider the problem of the capture cross section of a photon by a charged black hole. It is clear that the critical value of the impact parameter for a photon to be captured by a Reissner - Nordström black hole depends on the multiplicity root condition of the polynomial $R(r)$. This requirement is equivalent to the vanishing discriminant condition [31]. To find the critical value of impact parameter for Schwarzschild and RN metrics the condition has been used for corresponding cubic and quartic equations [32, 33, 34]. In particular, it was shown that for these cases the vanishing discriminant condition approach is more powerful in comparison with the procedure excluding r_{max} from the following system

$$R(r_{max}) = 0, \tag{6}$$

$$\frac{\partial R}{\partial r}(r_{max}) = 0, \tag{7}$$

as it was done, for example, by Chandrasekhar [29] (and earlier by Darwin [30]) to solve similar problems, because r_{max} is automatically excluded in the condition for vanishing discriminant.

Introducing the notation $\xi^2 = l, \mathcal{Q}^2 = q$, we obtain

$$R(r) = r^4 - lr^2 + 2lr - ql. \tag{8}$$

We remind basic algebraic definitions and relations.

If we consider an arbitrary polynomial $f(X)$ with degree n

$$f(X) = X^n + a_1X^{n-1} + ... + a_{n-1}X + a_n, \tag{9}$$

the elementary symmetric polynomials s_k have the following form, where $X_1, ...X_n$ are roots of the polynomial $f(X)$ [31]

$$s_k(X_1, ...X_n) = \sum_{1\leqslant i_1<i_2<...<i_k\leqslant n} X_{i_1} X_{i_2} ... X_{i_k}, \tag{10}$$

where $k = 1, 2, ..., n$.

The symmetrical k-power sum polynomial p_k have the following expression [31]

$$p_k(X_1, ...X_n) = X_1^k + X_2^k + ... + X_n^n, \quad \text{for} \quad k \geq 0. \tag{11}$$

To express p_k through s_k one can use Newton's equations [31]

$$p_k - p_{k-1}s_1 + ... + (-1)^{k-1}p_1 s_{k-1} + (-1)^k k s_k = 0, \text{ for } 1 \leqslant k \leqslant n; \tag{12}$$

$$p_k - p_{k-1}s_1 + ... + (-1)^{n-1}p_{k-n+1}s_{n-1} + (-1)^n p_{k-n}s_n = 0, \text{ for } k > n. \tag{13}$$

We introduce the following polynomial

$$\Delta_n(X_1, ...X_n) = \prod_{1\leqslant i<j\leqslant n} (X_i - X_j), \tag{14}$$

which can be represented as the Vandermonde determinant

$$\Delta_n(X_1, ...X_n) = \begin{vmatrix} 1 & 1 & ... & 1 \\ X_1 & X_2 & ... & X_n \\ ... & ... & ... & ... \\ X_1^{n-1} & X_2^{n-1} & ... & X_n^{n-1} \end{vmatrix}. \tag{15}$$

According to the discriminant Dis definition we have the $Dis(s_1, ..., s_n)$ polynomial

$$Dis(s_1, ..., s_n) = \Delta_n^2(X_1, ...X_n) = \prod_{1\leqslant i<j\leqslant n} (X_i - X_j)^2, \tag{16}$$

one can find [31]

$$Dis(s_1, ...s_n) = \begin{vmatrix} n & p_1 & p_2 & ... & p_{n-1} \\ p_1 & p_2 & p_3 & ... & p_n \\ p_2 & p_3 & p_4 & ... & p_{n+1} \\ ... & ... & ... & ... & ... \\ p_{n-1} & p_n & p_{n+1} & ... & p_{2n-2} \end{vmatrix}. \tag{17}$$

Clearly, that the vanishing discriminant condition is equivalent to an existence of multiple roots among roots $X_1, ...X_n$.

We apply this technique for the quartic polynomial $R(r)$ in Eq. (8).

So that the symmetric k-power polynomials for $n = 4$ have the form

$$p_k = {X_1}^k + {X_2}^k + {X_3}^k + {X_4}^k, k \geq 0. \tag{18}$$

The symmetric elementary polynomials for $n = 4$ have the form

$$\begin{aligned}
s_1 &= X_1 + X_2 + X_3 + X_4,\\
s_2 &= X_1X_2 + X_1X_3 + X_1X_4 + X_2X_3 + X_2X_4 + X_3X_4,\\
s_3 &= X_1X_2X_3 + X_2X_3X_4 + X_2X_3X_4,\\
s_4 &= X_1X_2X_3X_4.
\end{aligned} \tag{19}$$

We calculate the discriminant of the family X_1, X_2, X_3, X_4

$$Dis(s_1, s_2, s_3, s_4) = \begin{vmatrix} 1 & 1 & 1 & 1 \\ X_1 & X_2 & X_3 & X_4 \\ {X_1}^2 & {X_2}^2 & {X_3}^2 & {X_4}^2 \\ {X_1}^3 & {X_2}^3 & {X_3}^3 & {X_4}^3 \end{vmatrix}^2 = \begin{vmatrix} 4 & p_1 & p_2 & p_3 \\ p_1 & p_2 & p_3 & p_4 \\ p_2 & p_3 & p_4 & p_5 \\ p_3 & p_4 & p_5 & p_6 \end{vmatrix} \tag{20}$$

Expressing the polynomials $p_k (1 \leq k \leq 6)$ in terms of the polynomials $s_k (1 \leq k \leq 4)$ and using Newton's equations we calculate the polynomials and discriminant of the family X_1, X_2, X_3, X_4 in roots of the polynomial $R(r)$; we obtain

$$\begin{aligned}
&p_1 = s_1 = 0, \quad p_2 = -2s_2, \quad p_3 = 3s_3,\\
&p_4 = 2{s_2}^2 - 4s_4, \quad p_5 = -5s_3s_2,\\
&p_6 = -2{s_2}^3 + 3{s_3}^2 + 6s_4s_2,
\end{aligned} \tag{21}$$

where $s_1 = 0, s_2 = -l, s_3 = -2l, s_4 = -ql$, corresponding to the polynomial $R(r)$ in Eq. (8).

The discriminant Dis of the polynomial $R(r)$ has the form:

$$\begin{aligned}
Dis(s_1, s_2, s_3, s_4) &= \begin{vmatrix} 4 & 0 & 2l & -6l \\ 0 & 2l & -6l & 2l(l+2q) \\ 2l & -6l & 2l(l+2q) & -10l^2 \\ -6l & 2l(l+2q) & -10l^2 & 2l^2(l+6+3q) \end{vmatrix} = \\
&= 16l^3[l^2(1-q) + l(-8q^2 + 36q - 27) - 16q^3].
\end{aligned} \tag{22}$$

The polynomial $R(r)$ thus has a multiple root if and only if

$$l^3[l^2(1-q) + l(-8q^2 + 36q - 27) - 16q^3] = 0. \tag{23}$$

Excluding the case $l = 0$, which corresponds to a multiple root at $r = 0$, we find that the polynomial $R(r)$ has a multiple root for $r \geq r_+$ if and only if

$$l^2(1-q) + l(-8q^2 + 36q - 27) - 16q^3 = 0. \tag{24}$$

If $q = 0$, we obtain the well-known result for a Schwarzschild black hole [23, 24, 35], $l = 27$, or $L_{cr} = 3\sqrt{3}$. If $q = 1$, then $l = 16$, or $L_{cr} = 4$, which also corresponds to numerical results given in [36].

The photon capture cross section for an extreme charged black hole turns out to be considerably smaller than the capture cross section of a Schwarzschild black hole. The critical value of the impact parameter, characterizing the capture cross section for a RN black hole, is determined by the equation

$$l_{\rm cr} = \frac{(8q^2 - 36q + 27) + \sqrt{D_1}}{2(1-q)}, \tag{25}$$

where $D_1 = (8q^2 - 36q + 27)^2 + 64q^3(1-q) = -512\left(q - \frac{9}{8}\right)$. It is clear from the last relation that there are circular unstable photon orbits only for $q \leq \frac{9}{8}$ (see also results in [11] about the same critical value).

Substituting Eq.(25) into the expression for the coefficients of the polynomial $R(r)$ it is easy to calculate the radius of the unstable circular photon orbit (which is the same as the minimum periastron distance). The orbit of a photon moving from infinity with the critical impact parameter, determined in accordance with Eq.(25) spirals into circular orbit.

To find a radius of photon unstable orbit we will solve Eq. (7) substituting $l_{\rm cr}$ in the relation. From trigonometric formula for roots of cubic equation we have

$$r_{\rm crit} = 2\sqrt{\frac{l_{\rm cr}}{6}}\cos\frac{\alpha}{3}, \tag{26}$$

where

$$\cos\alpha = -\sqrt{\frac{27}{2l_{\rm cr}}}, \tag{27}$$

As it was explained in [9] this leads to the formation of shadows described by the critical value of L_{cr} or, in other words, in the spherically symmetric case, shadows are circles with radii L_{cr}. Therefore, measuring the shadow size, one could evaluate the black hole charge in black hole mass units M.

In Fig. 1 shadow size is given as a function of charge (including possible tidal charge with a negative q and super-extreme charge $q > 1$). In Fig. 2 radius of last unstable orbit for photons as a function of q is given.

4. Consequences

4.1. A disappearance of shadows for naked singularities

In spite of the cosmic censorship hypothesis [39] that a singularity has to be shielded by a horizon, properties of naked singularities are a subject

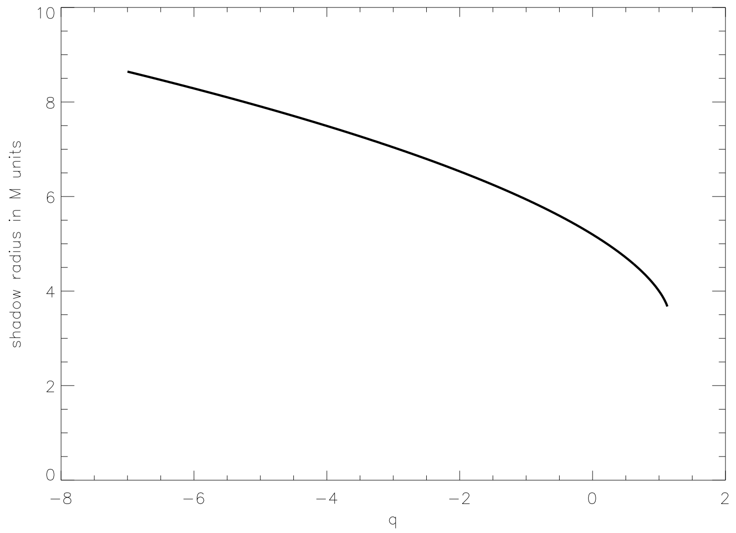

Figure 1: Shadow (mirage) sizes M units as a function of q.

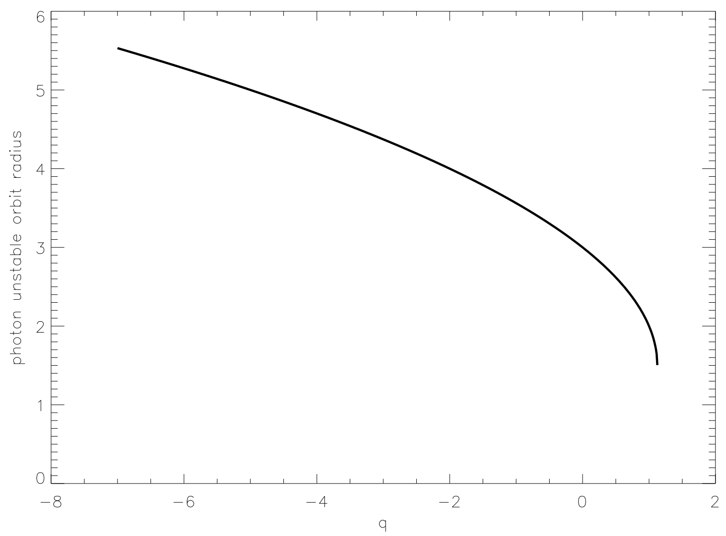

Figure 2: Radius of the last circular unstable photons orbit in M units as a function of q.

of intensive theoretical studies. As usual spherical symmetrical cases are more easier for analysis and RN metrics with super extreme charge $q > 1$ are investigated in a number of papers, see, for instance [40] and references therein.

So, if we assume that $q > 1$, we can see from Eq. (25) that for $q \leq 9/8$ we have shadows, while for $q > 9/8$ the shadows do not exist. For these charges ($q > 9/8$) incoming photons always scattering by black holes for $l \neq 0$ because the polynomial $R(r)$ has no multiple roots but it has a single positive root (it means scattering) since for great positive r we have $R(r) > 0$ while $R(0) < 0$. The degenerate case of radial trajectories of photons ($l = 0$) can be ignored as the case with "zero measure" or the structural unstable case using the Poincare – Pontryagin – Andronov – Anosov – Arnold terminology [37]. It means that in any small vicinity a behavior of other geodesics from the radial ones is qualitatively different, therefore, such objects can not be observed in nature. Therefore, shadows exist only for $q \leq 9/8$. So, $q = 9/8$ is critical value which is characterized "catastrophe" [38] or the qualitatively different behavior of the system (appearance and disappearance of shadows).

For the critical $q = 9/8$ we have the smallest shadow with $l = 27/2$ and a shadow size $\xi = \sqrt{13.5} \approx 3.674$ (in M units) or 84.38 μas in diameter for the black hole at the Galactic Center. For this impact parameter we have corresponding circular unstable orbit for photons with $r = 3/2$ (in M units).

4.2. Observational constraints on a charge of the black hole at the Galactic Center

If we adopt the distance toward the Galactic Center $d_* = 8.3$ kpc and mass of the black hole $M_{BH} = 4.4 * 10^6 M_\odot$ [41], then we have the angle 10.45 μas for the corresponding Schwarzschild radius $R_g = 2.95 * \dfrac{M_{BH}}{M_\odot} * 10^5$ cm, so a shadow size for the Schwarzschild black hole is around 54.2 μas, for a black hole with a tidal charge ($q = -6.4$) suggested by Bin-Nun [1, 2, 3] a shadow size is about 88.1 μas, while for the extreme charge ($q = 1$) and critical charge ($q = 9/8$) the shadow sizes are 41.8 μas and 38.4 μas, respectively.

4.3. Comparison with observations

A couple of year ago Doeleman et al. [5] claimed that intrinsic diameter of Sgr A^* is 37^{+16}_{-10} μas at the 3 σ confidence level. If we believe in GR and the central object is a black hole, then we have to conclude that a shadow is practically coincides with the intrinsic diameter, so in spite of the fact that a Schwarzschild black hole is marginally consistent with observations, a Reissner – Nordström black hole provides much better fit of a shadow size, while a black hole with a significant tidal charge ($q = -6.4$) is out of 9.6 σ level interval. Later on, an accuracy of intrinsic size measurements was significantly improved, so Fish et al. [42] gave $41.3^{+5.4}_{-4.3}$ μas (at 3 σ

level) on day 95, $44.4^{+3.0}_{-3.0}$ μas on day 96 and $42.6^{+3.1}_{-2.9}$ μas on day 97, so a tidal charge ($q = -6.4$) is out of 26 σ level for day 95 and even less probable for other observations.

The black hole in the elliptical galaxy M87 looks also perspective to evaluate shadow size [43] (probably even its shape in the future to estimate a black hole spin) because the distance toward the galaxy is 16 ± 0.6 Mpc [44], black hole mass is $M_{M87}(6.2 \pm 0.4) \times 10^9 M_\odot$ [45], so that an angle $(7.3 \pm 0.5)\mu as$ corresponds to the Schwarzschild radius [43], so the angle is comparable with the corresponding value considered earlier for our Galactic Center case. In paper [43] it was reported that smallest shadow size is $5.5 \pm 0.4 R_{SCH}$ with 1 σ errors (where $R_{SCH} = 2GM_{M87}/c^2$), so that at the moment the shadow size is consistent with the Schwarzschild metric for the object.

5. Conclusions

Based on observations [5, 42] one can say that between for the Schwarzschild black hole model we have tensions between evaluations of black hole mass done with observations of bright star orbits near the Galactic Center and the evaluated shadow size. To reduce tensions between estimates of the black hole mass and the intrinsic size measurements, one can use the Reissner – Nordström metric with a significant charge which is comparable with the critical one. We do not claim that the corresponding charge has an electric origin because an interstellar environment is electrically neutral, so the corresponding charge may be induced (like a tidal charge induced by extra dimension) and has a non-electric origin. Charge estimates for the Reissner – Nordström metric given from geodesic trajectories for orbital motions are given in [46].

Recent estimates of the smallest structure in the M87 published in paper [43] do not need an introduction of charge (tidal or normal) to fit observational data because sizes of the smallest spot near the black hole at the object are consistent with the shadow size evaluated for the Schwarzschild metric.

Acknowledgments

The author thanks D. Borka, V. Borka Jovanović, F. De Paolis, G. Ingrosso, P. Jovanović, A. A. Nucita, B. Vlahovic for useful discussions and B. Dragovich for his kind attention to this contribution.

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