Green-Clifford ansatz realization of Parabose representations*

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Abstract

Green's ansatz is a well known method for construction of "unique vacuum" representations of parabose (parafermi) algebra. Exploiting a Clifford algebra variant of the Green's ansatz we construct unitary representations with vacuum state carrying arbitrary $SU(n)$ representation (n being the number of parabose operator pairs).

1. Introduction

Parabose algebra was introduced by H.S. Green [1] long ago, as a generalization of the common bose algebra relations. Following the Green's definition, parabose algebra is algebra of n pairs of mutually hermitian conjugate operators $a_\alpha, a_\alpha^\dagger$, satisfying trilinear relations:

$$[\{a_\alpha, a_\beta^\dagger\}, a_\gamma] = -2\delta_{\beta\gamma} a_\alpha, \tag{1}$$

$$[\{a_\alpha, a_\beta\}, a_\gamma] = 0, \tag{2}$$

together with relations (additional four) that follow from these by hermitian conjugation and by use of Jacobi identities.[1]

In the same paper [1], Green offered a solution for the above relations, in the terms of sum of operators satisfying "mixed" commutation and anticommutation relations:

$$a_\alpha = \sum_{a=1}^{p} a_\alpha^a, \tag{3}$$

where a_α^a and $a_\alpha^{a\dagger}$ anticommute for different values of Green's indices a and b:

$$a \neq b \Rightarrow \{a_\alpha^{a\dagger}, a_\alpha^{b\dagger}\} = \{a_\alpha^a, a_\alpha^b\} = \{a_\alpha^a, a_\alpha^{b\dagger}\} = 0 \tag{4}$$

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[1] We note that, in a Hilbert space equipped with positive definite metrics (with respect to which one defines the adjoint $a_\alpha^\dagger$), all algebra relations actually follow from the single relation (1).

and behave as usual bose creation and annihilation operators otherwise:

$$a = b \Rightarrow [a^a_\alpha, a^{b\dagger}_\beta] = \delta_{\beta\alpha},\ [a^a_\alpha, a^b_\beta] = 0. \tag{5}$$

This construction is nowadays known as the "Green's ansatz" while the integer p is called the "order of parastatistics". Obviously, the case $p = 1$ corresponds to usual bose algebra.

Parabose algebra was originally introduced as an alternative, i.e. generalized method for field quantization, that would correspond to hypothetical particles obeying neither the usual Fermi nor the Bose statistics, but a generalization called parastatistics. In [3] Greenberg and Messiah have concluded that, in this particular context of parastatistics, Green's ansatz suffices for construction of all relevant unitary representations. However, his considerations included two elements special to the parastatistics context: 1) assumption of infinite many degrees of freedom n (i.e. infinite number parabose pairs) that effectively precluded solutions with non-integer p values, and 2) assumption that only "unique vacuum" Fock space representations are of physical interest.

On the other hand, parabose algebra has importance as an algebraic structure in its own right, irrespectively of the parastatistical context. Mathematically, it was realised by Ganchev and Palev [2] that this algebra is equivalent to the orthosymplectic $osp(1|2n)$ superalgebra.[2] In the light of this connection, parabose algebra, or, in other words, $osp(1|2n)$ superalgebra, has its importance in many physical areas/models. Of particular interest are the models where parabose algebra ($osp(1|2n)$ superalgebra) represents the space-time supersymmetry algebra (eg. [4, 5]). In this context number of parabose pairs n depends on the dimensionality of the space-time and ranges usually from $n = 4$ (in the four dimensional case) to $n = 32$ or $n = 64$ (string theory). The assumptions taken by Greenberg and Messiah in [3] here have no longer physical sense: 1) due to finite n, the order of parastatistics p can also take noninteger values from a certain continuum range (p_0, ∞), where p_0 is related to the, so called, first reduction point (Verma module terminology) [6], and 2) Fock vacuum has no more interpretation of "no particle state" but merely represents lowest conformal energy state and thus representations other than "unique vacuum" ones must also be considered. Being not applicable to both of these classes of representations, the basic form of the Green's ansatz construction is therefore no longer sufficient. Of the two, the latter shortcoming is far more serious. Namely, in the context of space-time symmetry, unitary irreducible representations (UIR's) of parabose algebra should be directly related to particle content of the model. Whereas it could be argued that non integer values of order of parastatistics p could be nonphysical, it is not so for the "unique vacuum" representations. On the contrary, in these

[2] This is exactly so if the parabose algebra is defined solely by structural relations, without any mention of Hermitian conjugation. However, if the algebra is introduced as in [1], then, strictly speaking, it is one concrete realization of the $osp(1|2n)$ superalgebra that has only positive energy unitary representations.

representations lowest weight state (i.e. Fock vacuum state) carries nontrivial representation of $SU(n)$ compact subgroup of $osp(1|2n)$ and thus they carry additional quantum numbers – the fact that makes corresponding particles physically highly interesting and important.

2. Unitary representations

In this section we recapitulate [7] classification of unitary irreducible representations of parabose algebra, as defined by (1,2) (results correspond to classification of positive energy UIR's of $osp(1|2n)$). The results were obtained by computer analysis of the lowest weight Verma module structure for cases $n \leq 4$, followed by a straightforward conjecture for the classification for case of arbitrary n.

First we will fix the notation and definitions, which basically follow that of [6].

We consider lowest weight Verma modules $V^\Lambda \cong U(\mathcal{G}^+) \otimes |v_0\rangle$. Here, $\mathcal{G}^+$ denotes subalgebra of positive roots in standard algebra decomposition $\mathcal{G}^\mathbb{C} = \mathcal{G}^+ \oplus \mathcal{H} \oplus \mathcal{G}^-$ ($\mathcal{G}$ denotes superalgebra $osp(1|2n)$ and $\mathcal{G}^\mathbb{C}$ its complexification; $\mathcal{H}$ is Cartan subalgebra) and $|v_0\rangle$ is a lowest weight vector of weight Λ:

$$X \in \mathcal{G}^- \Rightarrow X|v_0\rangle = 0, \quad H \in \mathcal{H} \Rightarrow H|v_0\rangle = \Lambda(H)|v_0\rangle. \tag{6}$$

Roots, expressed using elementary functionals, are:

$$\begin{aligned} \Delta &= \{\pm\delta_\alpha, 1 \leq \alpha \leq n; \pm\delta_\alpha \pm \delta_\beta, 1 \leq \alpha < \beta \leq n; \\ &\quad \pm 2\delta_\alpha, 1 \leq \alpha \leq n\} \end{aligned} \tag{7}$$

(the two signs in $\pm\delta_\alpha \pm \delta_\beta$ not being correlated) and the corresponding root vectors we will denote as (in the same order):

$$\begin{aligned} \mathcal{G}^+ \oplus \mathcal{G}^- &= \{a^\dagger_{\pm\alpha}, 1 \leq \alpha \leq n; a^\dagger_{\pm\alpha,\pm\beta}, 1 \leq \alpha < \beta \leq n; \\ &\quad a^\dagger_{\pm\alpha,\pm\alpha}, 1 \leq \alpha \leq n\}. \end{aligned} \tag{8}$$

Here we introduced a compact notation for superalgebra elements, that emphasises the parabose connection:

$$a^\dagger_{-\alpha} \equiv a_\alpha, \quad a^\dagger_{\alpha,\beta} \equiv \{a^\dagger_\alpha, a^\dagger_\beta\}. \tag{9}$$

Simple root vectors are:

$$\{a^\dagger_{-2,1}, a^\dagger_{-3,2}, \ldots, a^\dagger_{-n,n-1}, a^\dagger_n\}. \tag{10}$$

We will label representations by the signature

$$\chi = \{s_1, s_2, \ldots, s_{n-1}, d\}, \tag{11}$$

that is connected to the lowest weight Λ in the following way:

$$(\Lambda, \delta_\alpha) = d + \frac{1}{2}(a_1 + a_2 + \cdots + a_{\alpha-1} - a_\alpha - \cdots - a_{n-1}). \tag{12}$$

Notice that parameters $s_1, s_2, \ldots, s_{n-1}$ define behaviour of the lowest weight state $|v_0\rangle$ under action of the $SU(n)$ subgroup generated by elements $\{a_\alpha, a^\dagger_\beta\}$. The case $s_1 = s_2 = \cdots = s_{n-1} = 0$ corresponds to the "unique vacuum" representations.

We introduce a (Shapovalov) norm on the Verma module via natural involutive antiautomorphism: $\omega : \omega(a_\alpha) = a^\dagger_\alpha$ (compatible with the assumed Hilbert space metric). Right away we note that simple unitarity considerations – calculating norms of vectors $a^\dagger_{-(\alpha+1),\alpha}|v_0\rangle$ and $a^\dagger_1|v_0\rangle$ – result in constraints: $s_\alpha \geq 0$, $d \geq (s_1+s_2+\cdots+s_{n-1})/2$. Parameters $s_1, s_2, \ldots, s_{n-1}$ must be integers, labelling an $SU(n)$ Young tableau with $s_1+s_2+\cdots+s_{n-1}$ boxes in the first row, $s_1 + s_2 + \cdots + s_{n-2}$ boxes in the second and so on, ending with s_1 boxes in the row $n-1$.

For certain values of Λ, submodules appear in the structure of the Verma module V^Λ and the module becomes reducible. Basic case is when this happens due to existence of a singular vector $|v_s\rangle \in V^\Lambda$:

$$X|v_s\rangle = 0, \qquad \forall X \in \mathcal{G}^-. \tag{13}$$

This singular vector, in turn, generates a submodule $V^{\Lambda'} \cong U(\mathcal{G}^+)|v_s\rangle$ within V^Λ.

To ensure irreducibility, all submodules corresponding to singular vectors must be factored out. However, after factoring out these submodules, new singular vectors may appear in the remaining space – called subsingular vectors. Namely, if the union of all submodules of singular vectors is denoted by $\tilde{I}^\Lambda$ then a vector $|v_{ss}\rangle \in V^\Lambda$ is called a subsingular vector [9] if $|v_{ss}\rangle \notin \tilde{I}^\Lambda$ and:

$$X|v_{ss}\rangle \in \tilde{I}^\Lambda, \qquad \forall X \in \mathcal{G}^-. \tag{14}$$

Just as singular vectors, subsingular vectors also generate submodules that have to be factored out when looking for irreducible representations.

In the particular case of $osp(1|2n)$ there are always, irrespectively of d value, singular vectors of the form:

$$|v^\alpha_s\rangle \equiv (a^\dagger_{-(\alpha+1),\alpha})^{s_\alpha+1}|v_0\rangle, \quad \alpha = 1, 2, \ldots n-1, \tag{15}$$

(when considering cases of unitary and therefore finite dimensional $SU(n)$ representations μ, related to integer values of s_α). Of special interest thus are additional d-dependant singular vectors.

Our analysis of the Verma module structure heavily relied on the computer analysis and was carried out in the following general manner (that we just briefly describe). First, Kac determinant of a sufficiently high level was considered as a function of parameter d (for each given class of $SU(4)$ representation μ). In this way it was possible to locate the highest value of d for which the determinant vanishes and the Verma module becomes reducible. The singular or subsingular vector responsible for the singularity of the Kac matrix was then calculated, effectively by solving an (optimized) system of linear equations. Next we would find the norm of this vector and look for possible additional discrete reduction points at (lower) values of d for which the norm also vanishes. If new reduction points with new (sub)singular vectors were found it was also necessary to check that, upon removal of the corresponding submodules, no vectors with zero or negative norm remained. For this, it was enough to check that previously found (sub)singular vectors (i.e. those occurring for higher d values) belonged to the factored-out submodules. Optimized Wolfram Mathematica code was written to perform all these calculations. The analysis was carried out for $n \leq 4$ cases and the results turned out to be readily generalizable to the case arbitrary n. Classification of parabose UIR's is given in the following list, where the allowed values of the parameter d are given for different possible cases of parameters $s_1, s_2, \ldots, s_{n-1}$ values:

- $s_1 = s_2 = \cdots = s_{n-1} = 0$, i.e. "unique vacuum" UIR's:

$$\begin{array}{ll} d > (n-1)/2; & \\ d = (n-1)/2, & |v_{ss}^{(1,1,1,\ldots,1,1,1,1)}\rangle; \\ d = (n-2)/2, & |v_{ss}^{(0,1,1,\ldots,1,1,1,1)}\rangle; \\ & \cdots \\ d = 2/2, & |v_{ss}^{(0,0,0,\ldots,0,1,1,1)}\rangle; \\ d = 1/2, & |v_{s}^{(0,0,0,\ldots,0,0,1,1)}\rangle; \\ d = 0/2, & |v_{s}^{(0,0,0,\ldots,0,0,0,1)}\rangle; \end{array} \tag{16}$$

- $s_1 = s_2 = \cdots = s_{n-2} = 0, s_{n-1} > 0$, i.e. single row tableaux UIR's:

$$\begin{array}{ll} d > s_{n-1}/2 + (n-1+1)/2; & \\ d = s_{n-1}/2 + (n-1)/2, & |v_{ss}^{(1,1,1,\ldots,1,1,0)}\rangle; \\ d = s_{n-1}/2 + (n-1-1)/2, & |v_{ss}^{(0,1,1,\ldots,1,1,0)}\rangle; \\ & \cdots \\ d = s_{n-1}/2 + 4/2, & |v_{ss}^{(0,0,\ldots,1,1,1,0)}\rangle; \\ d = s_{n-1}/2 + 3/2, & |v_{s}^{(0,0,\ldots,0,1,1,0)}\rangle; \\ d = s_{n-1}/2 + 2/2, & |v_{s}^{(0,0,\ldots,0,0,1,0)}\rangle; \end{array} \tag{17}$$

- ...
- $s_1 = 0, s_2 > 0$, i.e. $(n-2)$ rows tableaux UIR's:

$$\begin{array}{ll} d > (s_2 + \cdots + s_{n-1})/2 + n - 3/2; & \\ d = (s_2 + \cdots + s_{n-1})/2 + n - 3/2, & |v_{s}^{(1,1,0,\ldots,0,0,0)}\rangle; \\ d = (s_2 + \cdots + s_{n-1})/2 + n - 4/2, & |v_{s}^{(0,1,0,\ldots,0,0,0)}\rangle; \end{array} \tag{18}$$

- $s_1 > 0$, i.e. $(n-1)$ rows tableaux UIR's:

$$\begin{array}{ll} d > (s_1 + \cdots + s_{n-1})/2 + n - 1; & \\ d = (s_1 + \cdots + s_{n-1})/2 + n - 1, & |v_s^{(1,0,0,\ldots,0,0,0)}\rangle. \end{array} \tag{19}$$

For each allowed value of d, existence of a corresponding singular or subsingular vector is indicated, using the following notation: ss in the lower index stands for "subsingular" whereas s means "singular" vector; in the upper index we give "relative weight" of the vector – if the (sub)singular vector generates Verma submodule of weight Λ' the the relative weight is $\Lambda' - \Lambda$. For UIR's from continuous d range, no (sub)singular vectors appear.

3. Construction of parabose UIR's

In this section we will use a Clifford algebra variant of Green's ansatz, first proposed by Greenberg and Macrae [8], to explicitly construct the listed parabose UIR's. Note, that, whereas Greenberg and Messieah have discussed use of Green's ansatz only for construction of "unique vacuum" UIR's [3], we will demonstrate that Green's ansatz suffices for construction of all discrete UIR's.

The method cannot be applied to UIR's from the continuous spectre, i.e. those UIR's that occur for non (half)integer values of parameter d. However, from the physical viewpoint, representations from the discrete spectre (d taking discrete (half)integer values less or equal to the first reduction point) are of greater significance since only in these cases singular or subsingular vectors appear. It is well known that these vectors turn into important equations of motion (e.g. see [9]). In the particular case of the parabose generalization of supersymmetry, these vectors, for example, turn into Klein-Gordon, Dirac and Maxwell equations [5].

In the same paper where he first introduced parabose (and parafermi) algebra [1], H.S.Green has also offered a way to construct some of the unitary representations using what is nowadays known as the Green's ansatz (3). Greenberg and Macrea in [8] introduced a "gauge-invariant" variant of the Green's ansatz, representing the annihilation parabose operators as the following sum:

$$a_\alpha = \sum_{a=1}^{p} a_\alpha^a \ e_a. \tag{20}$$

In this expression integer p is the order of the parastatistics, e_a are elements of a real Clifford algebra[3]:

$$\{e_a, e_b\} = 2\delta_{ab} \tag{21}$$

and operators a_α^a together with adjoint $a_\alpha^{a\dagger}$ satisfy ordinary bosonic algebra relations. There are total of $n \cdot p$ mutually commuting pairs of annihilation-creation operators $(a_\alpha^a, a_\alpha^{a\dagger})$:

$$[a_\alpha^a, a_\beta^{b\dagger}] = \delta_{\beta\alpha}\delta^{ab}; \quad [a_\alpha^a, a_\beta^b] = 0, \tag{22}$$

[3]Greenberg has also considered using complex Clifford algebra instead of real one, but that case requires altering of parabose algebra relations.

where $a, b = 1, 2, \dots p$ and $\alpha, \beta = 1, 2, \dots n$.

The overall Green's ansatz representation space of order p can be seen as tensor product of p multiples of Hilbert spaces $\mathcal{H}_a$ of ordinary linear harmonic oscillator in n-dimensions multiplied by the representation space of matrix representation of the Clifford algebra:

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_p \otimes \mathbb{C}^{2^{[p/2]}}. \tag{23}$$

A single factor Hilbert space $\mathcal{H}_a$ is the space of unitary representation of n dimensional bose algebra of operators $(a^a_\alpha, a^{a\dagger}_\alpha), \alpha = 1, 2, \dots n$: $\mathcal{H}_a \cong U(a^{a\dagger}_\alpha)|0\rangle_a$, where $|0\rangle_a$ is the usual Fock vacuum of factor space $\mathcal{H}_a$.

It is clear that no negative or zero norm states appear in this space. Therefore, if we can find, in this framework, a lowest weight vector $|v_0\rangle$ of a proper weight (corresponding to UIR's classified in the previous section) then the vectors of the form $\mathcal{P}(X)|v_0\rangle, \mathcal{P}(X) \in U(\mathcal{G}^+)$ will span that representation space.

The unique vacuum representations of order p are constructed upon lowest weight vector $|v_0^{\{0,\dots,0,d\}}\rangle$ of the form:

$$|0_p, w_0\rangle \equiv |0\rangle_1 \otimes |0\rangle_2 \otimes \cdots \otimes |0\rangle_p \otimes w_0, \tag{24}$$

where w_0 is an arbitrary (column) vector from $\mathbb{C}^{2^{[p/2]}}$ of unit norm (scalar product in $\mathbb{C}^{2^{[p/2]}}$ is defined in usual way). All representations with (half) integer d from the class (17) of the UIR's classification can be constructed in this manner. The order of parastatistics has, for this class, the following connection with the UIR signature: $p = 2d$.

However, construction of the "unique vacuum", i.e. $s_1 = s_2 = \cdots = s_{n-1} = 0$ representations within Green's ansatz was known already to Green and Greenberg [1, 3]. The nontrivial part is construction of other representations, in which the lowest weight state carries nontrivial representation of the $SU(n)$ subgroup. A key step toward this end is a specific "pairing" of factor spaces. We define operators:

$$A^{(k)\dagger}_{\alpha\pm} \equiv \tfrac{1}{\sqrt{2}}(a^{2k-1\dagger}_\alpha \pm ie_{(k)}a^{2k\dagger}_\alpha), \tag{25}$$

where $e_{(k)} \equiv -ie_{2k-1}e_{2k}$ are mutually commuting ($[e_{(k)}, e_{(l)}] = 0$) and hermitian, and, by a convention, Green's index put in brackets enumerates "pairs" of factors spaces.

We note the following important relations satisfied by the operators (25):

$$[A^{(k)}_{\alpha\pm}, A^{(l)\dagger}_{\beta\pm}] = \delta^{kl}\delta_{\alpha\beta}, \qquad [A^{(k)}_{\alpha\pm}, A^{(l)\dagger}_{\beta\mp}] = 0, \tag{26}$$

where $A^{(k)}_{\alpha\pm} = (A^{(k)\dagger}_{\alpha\pm})^\dagger = \frac{1}{\sqrt{2}}(a^{2k-1}_\alpha \mp ie_{(k)}a^{2k}_\alpha)$. In other words, operators $(A^{(k)\dagger}_{\alpha+}, A^{(k)}_{\alpha+})$ and $(A^{(k)\dagger}_{\alpha-}, A^{(k)}_{\alpha-})$ are two independent sets of bose creation-annihilation operators.

Expressed using these operators, the parabose operator a_α has the following form:

$$a_\alpha = \sum_{k=1}^{[p/2]} \sqrt{2} e_{2k-1} A^{(k)}_{\alpha -} + \epsilon\, e_p a^p_\alpha, \tag{27}$$

where $\epsilon \equiv (p \mod 2)$. This form directly follows from the definitions (20, 25) and relation $e_{2k-1} A^{(k)}_{\alpha\pm} e_{2k-1} = A^{(k)}_{\alpha\mp}$. The last term is simply a remainder left after the pairing, that exists when p is odd.

From (27) it immediately follows that parabose operators a_α will annihilate any state built by acting of $A^{(l)\dagger}_{\beta+}$ operators upon the Fock vacuum of order p:

$$a_\alpha \mathcal{P}(A^{(k)\dagger}_{\beta+})|0_p, w_0\rangle = 0, \tag{28}$$

with $\mathcal{P}(A^{(k)\dagger}_{\beta+})$ denoting arbitrary polynomial of the operators (25).

On the other hand, such states transform nontrivially under action of $SU(n)$ subgroup, which is readily seen from:

$$\{a^\dagger_\alpha, a_\beta\} = p\delta_{\alpha\beta} + 2\sum_{k=1}^{[p/2]} (A^{(k)\dagger}_{\alpha+} A^{(k)}_{\beta+} + A^{(k)\dagger}_{\alpha-} A^{(k)}_{\beta-}) + \epsilon\, 2 a^{p\dagger}_\alpha a^p_\beta. \tag{29}$$

It is this combination of properties that allows us to easily construct lowest weight states of non-unique vacuum representations by using operators (25).

The discrete UIR's that correspond to single row Young tableaux (17) are constructed upon the lowest weight state of the form (up to normalization):

$$|v_0^{\{0,\ldots,0,s_{n-1},d\}}\rangle \sim (A^{(1)\dagger}_{n+})^{s_{n-1}}|0_p, w_0\rangle, \tag{30}$$

where $p = 2d - s_{n-1}$. Note that such UIR's are obtainable for $p \geq 2$.

Those discrete UIR's corresponding to double rows Young tableaux are constructed by using antisymmetrized products of two $A^{(k)\dagger}_{\alpha+}$ operators:

$$|v_0^{\{0,\ldots,0,s_{n-2},s_{n-1},d\}}\rangle \sim (A^{(1)\dagger}_{n+} A^{(2)\dagger}_{n-1+} - A^{(1)\dagger}_{n-1+} A^{(2)\dagger}_{n+})^{s_{n-2}} (A^{(1)\dagger}_{n+})^{s_{n-1}}|0_p, w_0\rangle, \tag{31}$$

where $p = 2d - s_{n-2} - s_{n-1}$. Such UIR's are obtainable for $p \geq 4$, that is, at least two pairs of factor spaces are needed.

Construction of UIR's that correspond to Young tableaux with more rows follows the same obvious pattern.

By inspecting the classification of parabose UIR's (16-19) it is evident that all representations for which $2d \in \mathcal{N}$ can be constructed in this manner, in particular all representations corresponding to appearance of additional (sub)singular vectors.

4. On a symmetry of the ansatz

We note that this Clifford variant of the Green's ansatz possesses an intrinsic $SO(p)$ symmetry generated by the following hermitian operators:

$$G^{ab} \equiv \sum_{\alpha=1}^{n} i(a_\alpha^{a\dagger} a_\alpha^b - a_\alpha^{b\dagger} a_\alpha^a) + \frac{i}{4}[e^a, e^b], \tag{32}$$

where $e^a = e_a$. Note that two terms in (32) resemble orbital and spin parts of rotation generators and that all vectors from the space (23) of Green's ansatz belong to spinorial representations of this symmetry group. These generators commute with entire parabose algebra:

$$[G^{ab}, a_\alpha] = 0, \tag{33}$$

and this fact can help to solve problem of the reducibility of the Green's ansatz space (23) for a given p. Namely, due to this commutativity, all states from (23) are, apart from $osp(1|2n)$ quantum numbers, also labelled by quantum numbers of some (spinorial) UIR of $SO(p)$. Besides, behaviour of the vectors from (23) under action of $SO(p)$ group (32) is determined solely by transformation properties of the corresponding lowest weight vector $|v_0\rangle$. This is easily seen as all vectors belonging to a parabose UIR determined by the lowest weight vector $|v_0\rangle$ can be written as

$$\mathcal{P}(X)|v_0\rangle, \mathcal{P}(X) \in U(\mathcal{G}^+), \tag{34}$$

while

$$G^{ab}\mathcal{P}(X)|v_0\rangle = \mathcal{P}(X)G^{ab}|v_0\rangle. \tag{35}$$

With a suitable choice of positive root system of the $so(p)$ algebra, it can be shown that $osp(1|2n)$ lowest weight vectors of the form (30-31) are, at the same time, the highest (lowest) weight vectors of certain $SO(p)$ UIR's.

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