Parafermions and homotopy algebras

Todor Popov[*]
Institute for Nuclear Research and Nuclear Energy,
Bulgarian Academy of Sciences,
72 Tsarigradsko chaussée, 1784 Sofia, BULGARIA

ABSTRACT

We explore the Fock spaces of the parafermionic algebra closed by the creation and annihilation operators introduced by H.S. Green. Each parafermionic Fock space allows for a free minimal resolution by graded modules of the graded 2-step nilpotent subalgebra of the parafermionic creation operators. Such a free resolution is constructed with the help of a classical Kostant's theorem computing suitable Lie algebra cohomologies (of the creation nilpotent subalgebra with values in the parafermionic Fock space). We give a cohomological interpretation of the Schur functions identities which have been recently discovered by Stoilova and Van der Jeugt. The endomorphisms of the parafermionic minimal free resolution close a differential graded algebra which is naturally endowed with the structure of a Stasheff homotopy algebra.

1. Parastatistics Algebras

H.S. Green introduced a scheme of quantization based on algebras of non-canonical commutation relations between the creation and annihilation operators. A quantum system obeying the Fermi-Dirac statistics and usually quantized according to the canonical anticommutation relations was quantized by H.S. Green with the help of the Lie algebra coined *parafermionic algebra* [6]

$$\begin{array}{rclrcl} [[a_i^\dagger, a_j], a_k^\dagger] &=& 2\delta_{jk} a_i^\dagger \,, & [[a_i^\dagger, a_j], a_k] &=& -2\delta_{ik} a_j \,, \\ [[a_i^\dagger, a_j^\dagger], a_k^\dagger] &=& 0 \,, & [[a_i, a_j], a_k] &=& 0 \,. \end{array} \tag{1}$$

Such particles are coined parafermions. Similarly one can introduce parabosons, replacing the canonical commutation relations of the Bose-Einstein statistics with the *parabosonic algebra*, which is the Lie super algebra (1) in which the Lie brackets are understand as Lie super-brackets. Parafermionic and parabosonic algebras are called parastatistics algebras.

[*] e-mail address: tpopov@inrne.bas.bg

The parafermionic algebra $\mathfrak{g}$ with finite number degrees of freedom n is isomorphic to a semi-simple Lie algebra

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta_+} \mathfrak{g}_\alpha \oplus \bigoplus_{\alpha\in\Delta_-} \mathfrak{g}_\alpha \ , \qquad (2)$$

for a root system $\Delta = \Delta_+ \cup \Delta_-$ of type B_n with positive roots Δ_+ given by

$$\Delta_+ = \{e_i\}_{1\leq i\leq n} \cup \{e_i + e_j, e_i - e_j\}_{1\leq i<j\leq n} \ , \quad \text{and} \quad \Delta_- = -\Delta_+ \ .$$

Hence the parafermionic algebra $\mathfrak{g}$ is the orthogonal algebra $\mathfrak{g} \cong \mathfrak{so}_{2n+1}$ written into an oscillator physical Cartan-Weyl basis $a_i^\dagger := E^{e_i}$ and $a_i := E^{-e_i}$ endowed with the anti-involution $\dagger$.

In the same vein the parabosonic algebra $\tilde{\mathfrak{g}}$ with finite number degrees of freedom n is isomorphic to the Lie superalgebra of type $B_{0,n}$ in the Kac table, *i.e.,* the orthosymplectic algebra $\tilde{\mathfrak{g}} \cong \mathfrak{osp}_{1,2n}$. More generally a system with n parabosonic and m parafermionic degrees of freedom is quantized [17] by the Lie superalgebra of type $B_{m,n}$, that is, $\mathfrak{osp}_{1+2m,2n}$.

Note that the $\mathbb{Z}_2$-grading of the parastatistics (super)-algebras is the opposite to the canonical one, in which bosons are even and fermions are odd generators.

In this note we will stick to the case of the parafermionic algebra $\mathfrak{g} = \mathfrak{so}_{2n+1}$.

2. Parafermionic Fock space

The parafermionic Lie algebra $\mathfrak{g}$ has decomposition $\mathfrak{g} = \mathfrak{n}^* \rtimes \mathfrak{u} \ltimes \mathfrak{n}$, where $\mathfrak{u}$ is a reductive and $\mathfrak{n}$ is a nilpotent Lie algebra. The creation operators $a_i^\dagger$ generate the free 2-step nilpotent Lie algebra $\mathfrak{n}$. The algebra $\mathfrak{n}$ is positively graded with degrees $\mathfrak{n}_1 = \bigoplus_i \mathbb{C}a_i^\dagger =: V$ and $\mathfrak{n}_2 = [\mathfrak{n}_1, \mathfrak{n}_1] \cong \wedge^2 V$ (and $\mathfrak{n}_i = 0$ for $i \geq 3$). Likewise $\mathfrak{n}^*$ is generated by the annihilation operators a_i. The parafermionic relations (1) imply that the generators $E_{ij} = \frac{1}{2}[a_i^\dagger, a_j]$ close the real form $\mathfrak{u}_n$ of a linear algebra $\mathfrak{gl}_n$ due to $E_{ij}^\dagger = E_{ji}$ and

$$[E_{ij}, E_{kl}] = \delta_{jk}E_{il} - \delta_{il}E_{jk} \ .$$

The vector space $V = \mathfrak{n}_1$ is the fundamental representation for $\mathfrak{gl}$, $E_{ij}a_k^\dagger = \delta_{jk}a_i^\dagger$.

The parabolic subalgebra $\mathfrak{p} = \mathfrak{n}^* \rtimes \mathfrak{gl} \subset \mathfrak{g}$ acts trivially on the 1-dimensional representation spanned on the vacuum state $|0\rangle$. The induced representation of the vacuum module is isomorphic the universal enveloping algebra[1] $U\mathfrak{n}$ [3] generated by the free action of the creation operators

$$\mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}}\mathbb{C}|0\rangle = U\mathfrak{g} \otimes_{U\mathfrak{p}} \mathbb{C}|0\rangle \cong U\mathfrak{n} \ .$$

[1]The algebra $U\mathfrak{n}$ was denoted $PS(V)$ in the paper [13] as an abriviation of ParaStatistics

Definition 2.1. *The parafermionic Fock space $\mathcal{V}(p)$ is the unitary representation $\mathcal{V}(p)$ of the parafermionic algebra $\mathfrak{g} \cong \mathfrak{so}_{2n+1}$ built on a unique vacuum vector $|0\rangle$ such that*

$$a_i|0\rangle = 0\ , \qquad [a_i, a_j^\dagger]|0\rangle = p\delta_{ij}|0\rangle\ . \tag{3}$$

The non-negative integer p is called the order of the parastatistics.

The vacuum vector $|0\rangle$ is the lowest weight vector of a $\mathfrak{g}$-module with weight $-\frac{p}{2}\sum e_i$ which is annihilated by all annihilation (lowering) operators a_i. The parafermionic Fock space $\mathcal{V}(p)$ with an integer p is finite-dimensional and it has a unique Highest Weight (HW) vector

$$|\Lambda\rangle = (a_1^\dagger)^p \ldots (a_n^\dagger)^p|0\rangle\ , \tag{4}$$

thus the $\mathfrak{so}_{2n+1}$-module $\mathcal{V}(p)$ is a highest weight module of weight Λ

$$V^\Lambda = \mathcal{V}(p), \qquad \Lambda = \frac{p}{2}\sum_{i=1}^n e_i\ .$$

The parafermionic algebra of order $p = 1$ coincides with the ordinary fermionic Fock space. The physical meaning of the order p for the parafermionic algebra is the number of particles that can occupy same state, that is, we encounter Pauli exclusion principle of order p. The symmetric submodule $S^{p+1}\mathfrak{n}_1 \subset \mathfrak{n}_1^{\otimes p+1}$ is spanned by the "exclusion condition" $(a_i^\dagger)^{p+1} = 0$ and it generates a twosided ideal $(S^{p+1}\mathfrak{n}_1)$. The parafermionic Fock space $\mathcal{V}(p)$ as Lowest Weight module will be isomorphic to the factor module of $U\mathfrak{n}$ by the "exclusion" ideal $(S^{p+1}V)$

$$\mathcal{V}(p) \cong U\mathfrak{n}/(S^{p+1}\mathfrak{n}_1)\ .$$

Equivalently the parafermionic Fock space $\mathcal{V}(p) = V^\Lambda$ is a HW $\mathfrak{g}$-module with HW vector $|\Lambda\rangle$, cf. eq. (4)

$$V^\Lambda \cong U\mathfrak{n}^*/(S^{p+1}\mathfrak{n}_1^*)) = \mathcal{V}(p)\ .$$

2.2. Character of $\mathcal{V}(p)$

Weyl character formula applied for Schur module V^λ yields the Schur polynomial

$$s_\lambda(x_1,\ldots,x_n) = \sum_{w\in W_1}\varepsilon(w)e^{w(\rho_1+\lambda)}/\sum_{w\in W_1}\varepsilon(w)e^{w(\rho_1)}, \qquad W_1 := S_n\ ,$$

where the variables are $x_i := \exp(-e_i)$ and the vector $\rho_1 = \frac{1}{2}\sum_{i=1}^n(n-2i+1)e_i$. Alternatively the Schur polynomials is written as

$$s_\lambda(x_1,\ldots,n_n) = \frac{\det||x_j^{\rho_{1i}+\lambda_i}||}{\det||x_j^{\rho_{1i}}||}\ . \tag{5}$$

The sum over the Schur functions with no more than p columns can be represented as a quotient of determinants (see p.84 in the book of Macdonald [16])

$$D_{\rho+p\theta}/D_\rho = e^{p\theta} \sum_{\lambda:\, l(\lambda')\leq p} s_\lambda(x_1,\ldots,x_n)\,, \qquad \theta = \frac{1}{2}\sum e_i, \tag{6}$$

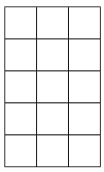
where $W = S_n \ltimes \mathbb{Z}_2^n$ is the Weyl group of the root system of Dynkin type B_n and $D_\rho = \sum_{w\in W}\varepsilon(w)e^{w\rho}$ with $\rho = \frac{1}{2}\sum_{i=1}^n(2n-2i+1)e_i$. Here λ' stands for the partition conjugated to λ and $l(\mu)$ is the length of the partition μ.

The LHS of the identity (6) can be interpreted [19] as a character of the parafermionic representation $\mathcal{V}(p)$ therefore the identity between $\mathfrak{so}_{2n+1}$ and $\mathfrak{gl}_n$ characters implies the following lemma.

Lemma 2.3. *The HW $\mathfrak{so}_{2n+1}$-module $V^\Lambda \cong \mathcal{V}(p)$ of HW vector $|\Lambda\rangle = |p\theta\rangle$ splits into a sum of irreducible $\mathfrak{gl}_n$-modules V^λ*

$$V^\Lambda \downarrow^{\mathfrak{so}_{2n+1}}_{\mathfrak{gl}_n} = \bigoplus_{\lambda:\lambda\subseteq(p^n)} V^\lambda\,,$$

where the sum runs over all partitions which match inside the $n\times p$ frame, e.g.



$$n=5, \quad p=3.$$

The vectors in the parafermionic Fock space $\mathcal{V}(p)$ are in bijection with the semi-standard Young Tableaux matching into the $n\times p$ rectangular frame [2, 12, 13]. An explicite basis for $\mathcal{V}(p)$ in terms of the Gel'fand-Zetlin patterns is given in [19].

We recall the Frobenius notation for a Young diagram η

$$\eta := (\alpha_1,\ldots,\alpha_r|\beta_1,\ldots,\beta_r) \qquad r = rank(\eta)$$

where the rank r is the number of boxes on the diagonal of η, arm-length α_i is the number of boxes on the right of the ith diagonal box, and leg-length β_i is the of boxes below the ith diagonal box. The number of boxes $|\eta|$ in η is

$$|\eta| = r + \sum_{i=1}^r \alpha_i + \sum_{i=1}^r \beta_i\,.$$

The conjugated diagram η' is the one in which the arms and legs are exchanged

$$\eta' := (\beta_1,\ldots,\beta_r|\alpha_1,\ldots,\alpha_r)\,.$$

It is worth noting that the set of self-conjugated Young diagrams $\{\lambda|\lambda = \lambda'\}$ is singled out by the condition $\beta_i = \alpha_i$ in Frobenius notation.

In the works [15] and [20] Lievens, Stoilova and Van der Jeugt conjectured sign-alternating character formulas for the parabosonic and parafermionic Fock spaces. In fact if one changes the category of vector spaces to the category of vector super-spaces [13] by functorial property the parabosonic formula will follow from the parafermionic one. So we will concentrate on the sign-alternating parafermionic character identity [20] for $\mathcal{V}(p)$

$$\frac{\sum_{\mu:\mu=\mu'}(-1)^{\frac{1}{2}(r+|\mu|)}s_{\mu^{(p)}}(x)}{\prod_i(1-x_i)\prod_{i<j}(1-x_ix_j)} = \sum_{\lambda:l(\lambda')\leq p} s_\lambda(x)\,. \tag{7}$$

Here the sum runs on the self-conjugated Young diagrams. The Young diagram $\mu^{(p)}$ is the arm p-augmentation of μ, which is the diagram such that the arm-lengths are augmented with p. Here μ is self-conjugated $\mu = (\alpha|\alpha)$ thus the arm p-augmentation $\mu^{(p)}$ reads

$$\mu^{(p)} = (\alpha_1 + p, \ldots, \alpha_r + p|\alpha_1, \ldots, \alpha_r). \tag{8}$$

The conjectured character identities has been proven by Ron King [10]. For a sketch of an alternative proof see [13].

3. Cohomology of $\mathfrak{n}$

We will give a homological interpretation of the sign-alternating character formula relying on the homology $H_\bullet(\mathfrak{n}, \mathcal{V}(p))$ and the cohomology $H^\bullet(\mathfrak{n}, \mathcal{V}(p))$ with coefficients in the parafermionic space $\mathcal{V}(p)$.

Recall that the standard Chevalley-Eilenberg complex of the Lie algebra $\mathfrak{n}$ is the chain complex $C_\bullet(\mathfrak{n}) = (U\mathfrak{n} \otimes_{\mathbb{K}} \wedge^k\mathfrak{n}, d_p)$ with differential

$$\begin{aligned} d_k(u\otimes x_1\wedge\ldots\wedge x_p) &= \sum_i(-1)^{i+1}ux_i\otimes x_1\wedge\ldots\wedge\hat{x}_i\wedge\ldots\wedge x_p \\ &+\sum_{i<j}(-1)^{i+j}u\otimes[x_i,x_j]\wedge x_1\wedge\ldots\wedge\hat{x}_i\wedge\ldots\wedge\hat{x}_j\wedge\ldots\wedge x_k\,. \end{aligned} \tag{9}$$

It provides a non-minimal projective(in fact free) resolution of $\mathbb{C}$, $C_\bullet(\mathfrak{n}) \stackrel{\epsilon}{\to} \mathbb{C}$. With the help of the standard resolution $C_\bullet(\mathfrak{n})$ the homologies $H_k(\mathfrak{n}, \mathbb{C})$ of the Lie algebra $\mathfrak{n}$ with trivial coefficients is obtained as the homology of the derived complex $\mathbb{C}\otimes_{U\mathfrak{n}} C_\bullet(\mathfrak{g})$

$$H_k(\mathfrak{n}, \mathbb{C}) = \mathrm{Tor}_k^{U\mathfrak{n}}(\mathbb{C}, \mathbb{C}) \cong H_k(\mathbb{C}\otimes_{U\mathfrak{n}} C_\bullet(\mathfrak{n}))\,.$$

The (right) derived complex $\mathbb{C}\otimes_{U\mathfrak{n}} C_\bullet(\mathfrak{n})$ is the chain complex with degrees $\bigwedge^\bullet\mathfrak{n} = \mathbb{C}\otimes_{U\mathfrak{n}} U\mathfrak{n}\otimes\bigwedge^\bullet\mathfrak{g}$ and differentials $\partial_k := id\otimes_{U\mathfrak{n}} d_k : \bigwedge^k\mathfrak{n} \to \bigwedge^{k-1}\mathfrak{n}$. The differential ∂ is induced by the Lie bracket $[\,\cdot\,,\,\cdot\,] : \bigwedge^2\mathfrak{n} \to \mathfrak{n}$ of the graded

Lie algebra $\mathfrak{n} = \mathfrak{n}_1 \oplus \mathfrak{n}_2$. It identifies a pair of degree 1 generators $a_i^\dagger, a_j^\dagger \in \mathfrak{n}_1$ with one degree 2 generator $a_{ij}^\dagger := (a_i^\dagger \wedge a_j^\dagger) = [a_i^\dagger, a_j^\dagger] \in \mathfrak{n}_2$. The differential ∂_k is the extension of the mapping $\partial_2 := -[\,\cdot\,,\,\cdot\,]$ on the exterior powers $\bigwedge^k \mathfrak{n}$. In greater details the chain complex has degrees

$$\bigwedge^k \mathfrak{n} = \bigwedge^k (V \oplus \wedge^2 V) = \bigoplus_{s+r=k} \wedge^s(\wedge^2 V) \otimes \wedge^r(V) \tag{10}$$

and differentials $\partial_k : \wedge^s(\wedge^2 V) \otimes \wedge^r(V) \to \wedge^{s+1}(\wedge^2 V) \otimes \wedge^{r-2}(V)$ such that

$$\begin{aligned} \partial_k : a_{i_1 j_1}^\dagger \wedge \ldots \wedge a_{i_s j_s}^\dagger \ \otimes \ a_1^\dagger \wedge \ldots \wedge a_r^\dagger \ \ \mapsto \\ \sum_{i<j} (-1)^{i+j} a_{ij}^\dagger \wedge a_{i_1 j_1}^\dagger \wedge \ldots \wedge a_{i_s j_s}^\dagger \ \otimes \ a_1^\dagger \wedge \ldots \wedge \hat{a}_i^\dagger \wedge \ldots \wedge \hat{a}_j^\dagger \wedge \ldots \wedge a_r^\dagger \, . \end{aligned}$$

More generally, the homology with coefficients in the right $U\mathfrak{n}$-module M is given by the homology of the derived complex $M \otimes_{U\mathfrak{n}} C_\bullet(\mathfrak{g})$

$$H_k(\mathfrak{n}, M) = \mathrm{Tor}_k^{U\mathfrak{n}}(M, \mathbb{C}) \cong H_k(M \otimes_{U\mathfrak{n}} C_\bullet(\mathfrak{n})) \, .$$

On the other hand the cohomology of the Lie algebra $\mathfrak{n}$ with coefficients in a left $U\mathfrak{n}$-module M' is obtained by the cohomology of the (left) derived complex $\mathrm{Hom}_{U\mathfrak{n}}(C_\bullet(\mathfrak{n}), M')$

$$H^k(\mathfrak{n}, M') = \mathrm{Ext}_{U\mathfrak{n}}^k(\mathbb{C}, M') = H^k(\mathrm{Hom}_{U\mathfrak{n}}(C_\bullet(\mathfrak{n}), M')) \, .$$

The left derived complex $\mathrm{Hom}_{U\mathfrak{n}}(C_\bullet(\mathfrak{n}), M')$ is a cochain complex. A p-cochain $f \in \wedge^k \mathfrak{n}^* \otimes M'$ is a multilinear function $f(x_1, \ldots, x_p)$ with variables $x_i \in \mathfrak{n}$ and values in the left $\mathfrak{n}$-module M'. The coboundary δf of such a $(k+1)$-cochain

$$\begin{aligned} \delta f(x_1, \ldots, x_{k+1}) = & \sum_i (-1)^{i+1} x_i f(x_1, \ldots, \hat{x}_i, \ldots, x_{k+1}) \\ & + \sum_{i<j} (-1)^{i+j} f([x_i, x_j], x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_{k+1}) \, . \end{aligned} \tag{11}$$

In particular, the cochain complex $\mathrm{Hom}_{U\mathfrak{n}}(C_\bullet(\mathfrak{n}), \mathbb{C}) = (\bigwedge^\bullet \mathfrak{n}^*, \delta)$ calculates the cohomology[2]

$$H^k(\mathfrak{n}, \mathbb{C}) = \mathrm{Ext}_{U\mathfrak{n}}^k(\mathbb{C}, \mathbb{C}) \cong H^k(\mathrm{Hom}_{U\mathfrak{n}}(C_\bullet(\mathfrak{g}), \mathbb{C})) \, . \tag{12}$$

The coboundary map $\delta^k : \bigwedge^k \mathfrak{n}^* \to \bigwedge^{k+1} \mathfrak{n}^*$ is transposed to the differential ∂_{k+1}

$$\begin{aligned} \delta^k : \ \ a_{i_1 j_1} \wedge \ldots \wedge a_{i_s j_s} \ \otimes \ a_1 \wedge \ldots \wedge a_r \ \ \mapsto \\ \sum_{l=1}^{s} (-1)^{i+j} a_{i_1 j_1} \wedge \ldots \wedge \hat{a}_{i_l j_l} \wedge \ldots \wedge a_{i_s j_s} \ \otimes \ a_{i_l} \wedge a_{j_l} \wedge a_1 \wedge \ldots \wedge \ldots \wedge a_r \, , \end{aligned} \tag{13}$$

[2] In the presence of metric one has $\delta := \partial^*$(see below)

where the sum is over $i_l < j_l$. The differential δ^k is (up to a conventional sign) the extension of the dualization of the Lie bracket $\delta^1 := [\,\cdot\,,\,\cdot\,]^* : \mathfrak{n}^* \to \bigwedge^2 \mathfrak{n}^*$ by the Leibniz rule(i.e., as derivation).

Since the algebra $\mathfrak{n}$ is nilpotent the universal enveloping algebra $U\mathfrak{n}$ is not only filtrated but also graded. Let us suppose that the right $U\mathfrak{n}$-module M is graded module. H. Cartan [1] proved that one has the identification between homology and cohomology of $\mathfrak{n}$ with coefficients in graded $U\mathfrak{n}$-modules

$$H_k(\mathfrak{n}, M)^* = \mathrm{Tor}_k^{U\mathfrak{n}}(M, \mathbb{C})^* \cong \mathrm{Ext}^k_{U\mathfrak{n}}(\mathbb{C}, M^*) = H^k(\mathfrak{n}, M^*) \qquad (14)$$

where the left $U\mathfrak{n}$-module $M^* = \mathrm{Hom}(M, \mathbb{C})$ is the graded-dual of the right $U\mathfrak{n}$-module M. In particular one has

$$H_k(\mathfrak{n}, \mathbb{C})^* = H^k(\mathfrak{n}, \mathbb{C}) \; . \qquad (15)$$

In our application we will take the left $U\mathfrak{n}$-module M' to be the parafermionic Fock space $V^\Lambda = \mathcal{V}(p)$ thus M^* will be the right $U\mathfrak{n}$-module $\mathcal{V}(p)^* = V^{*\,\Lambda}$ built at the top of the left vacuum $\langle 0| = |0\rangle^*$.

3.1. Kostant's theorem

The Kostant theorem is a powerfull tool which enable us to calculate cohomologies. Let's have a semi-simple algebra $\mathfrak{g}$ and its Borel subalgebra

$$\mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta_+} \mathfrak{g}_\alpha \; .$$

Any parabolic subalgebra $\mathfrak{p}$, $\mathfrak{g} \supset \mathfrak{p} \supseteq \mathfrak{b}$ has a Levi decomposition $\mathfrak{p} = \mathfrak{g}_1 \ltimes \mathfrak{n}$ where $\mathfrak{g}_1$ is a reductive algebra and $\mathfrak{n}$ is the nilradical (largest nilpotent ideal) of $\mathfrak{p}$. Consider the $\mathfrak{g}$-module V^Λ of weight Λ and the cohomology $H^\bullet(\mathfrak{n}, V^\Lambda)$ with coefficients in the reduction $\mathfrak{n}$-module $V^\Lambda \downarrow^{\mathfrak{g}}_{\mathfrak{n}}$. The Kostant's theorem gives the decomposition of $H^\bullet(\mathfrak{n}, V(\lambda))$ as a sum of irreducibles $\mathfrak{g}_1$-modules V^μ.

Theorem 3.2. *(Kostant) Let W be the Weyl group of the algebra $\mathfrak{g}$ and the subset $\Phi_\sigma \subseteq \Delta_+$ be*

$$\Phi_\sigma := \sigma\Delta_- \cap \Delta_+ \subseteq \Delta_+ \; .$$

Let ρ be the Weyl vector $\rho = \frac{1}{2}\sum_{\alpha\in\Delta_+} \alpha$. The roots of the nilpotent radical $\mathfrak{n}$ are denoted as $\Delta(\mathfrak{n})$ and the subset $W^1 = \{\sigma \in W | \Phi_\sigma \subset \Delta(\mathfrak{n})\}$ is a cross section of the coset $W_1 \backslash W$. The cohomology $H^\bullet(\mathfrak{n}, V^\Lambda)$ has a decomposition into irreducible $\mathfrak{g}_1$-modules V^μ

$$H^\bullet(\mathfrak{n}, V^\Lambda) = \bigoplus_{\sigma \in W^1} V^{\sigma(\rho+\Lambda)-\rho}$$

where the cohomological degree $H^j(\mathfrak{n})$ is given by the number of the elements $j := \#\Phi_\sigma$.

3.3. The cohomology $H^\bullet(\mathfrak{n},\mathbb{C})$

From our perspective the cohomology $H^\bullet(\mathfrak{n},\mathbb{C})$ with trivial coefficients is the special case of parafermionic Fock space with $p=0$ when $\mathcal{V}(p)=\mathbb{C}$. So we start by review of this simpler situation.

Theorem 3.4. *Let $\mathfrak{n}$ be the free 2-step nilpotent algebra $\mathfrak{n}=V\oplus\wedge^2V$. The cohomology $H^\bullet(\mathfrak{n},\mathbb{C})=\oplus_p H^p(\mathfrak{n},\mathbb{C})$ is a reducible $\mathfrak{gl}(V)$-module and decomposes into irreducible Schur modules $V^{*\mu}$ with self-conjugated Young diagrams*

$$H^k(\mathfrak{n},\mathbb{C})=\bigoplus_{\mu:\mu=\mu'} V^{*\mu}\,,\qquad\qquad k=\frac{1}{2}(|\mu|+r(\mu))\,.$$

Here the weight $|\mu|$ (the rank $r(\mu)$) is the number of boxes (on the diagonal) in μ.

Here we will follow closely an approach due to J. Grassberger, A. King and P. Tirao [5] which is an application of the Konstant's theorem.

Proof. Consider the parafermionic algebra $\mathfrak{g}\cong\mathfrak{so}_{2n+1}$ with Cartan decomposition (2). Consider the parabolic algebra $\mathfrak{p}\subset\mathfrak{g}$

$$\mathfrak{p}=\bigoplus_{i>j}\mathfrak{g}_{e_i-e_j}\oplus\mathfrak{h}\oplus\bigoplus_{\alpha\in\Delta_+}\mathfrak{g}_\alpha\,.$$

From the parafermionic relations (1) is readily seen that the Levi decomposition of the parabolic subalgebra $\mathfrak{p}=\mathfrak{g}_1\ltimes\mathfrak{n}$ has reductive component

$$\mathfrak{g}_1=\mathfrak{h}\oplus\bigoplus_{i\neq j}\mathfrak{g}_{e_i-e_j}\ \cong\mathfrak{gl}_n \tag{16}$$

acting by automorphisms on the free 2-step nilpotent algebra $\mathfrak{n}$ (the space $\mathfrak{n}_1=V$ being the fundamental representation of $\mathfrak{g}_1=\mathfrak{gl}_n$)

$$\mathfrak{n}\ \ =\ \ \bigoplus_i\mathfrak{g}_{e_i}\oplus\bigoplus_{i<j}\mathfrak{g}_{e_i+e_j}\cong V\oplus\wedge^2V\,. \tag{17}$$

The Weyl group W_1 of $\mathfrak{g}_1=\mathfrak{gl}_n$ is the symmetric group S_n operating on $\{e_1,\dots$ $\dots,e_n\}$ by permutations. The Weyl group of $\mathfrak{g}=\mathfrak{so}_{2n+1}$ is $W=S_n\ltimes\mathbb{Z}_2^n$. The $\mathbb{Z}_2^n$ is generated by operators τ_i, $i=1,\dots,n$ such that $\tau_i^2=1$ acting by

$$\tau_i(e_j)=\begin{cases}-e_j & i=j\\ \ \ e_j & i\neq j\end{cases}\,.$$

The elements $\tau_I\in\mathbb{Z}_2^n$ are indexed by subsets $I\subseteq\{1,\dots,n\}$, $\tau_I\in\prod_{i\in I}\tau_i$.

Let us describe the subset W^1 which has order $|W^1|=2^n$. Both W^1 and $\mathbb{Z}_2^n$ are cross sections of $W_1\backslash W$ thus for each $\tau_I\in\mathbb{Z}$ exists a unique permutation $\omega_I\in S_n$ such that $\omega_I\tau_I\in W^1$.

Let $\mathfrak{b}^0$ be the nilpotent part of the Borel algebra $\mathfrak{b}^0 = \mathfrak{b}/\mathfrak{h}$ and and the complement be $\mathfrak{m}_1 = \mathfrak{g}_1 \cap \mathfrak{b}^0 = \mathfrak{b}^0/\mathfrak{n}$. The subset $W^1 = \{\sigma \in W | \Phi_\sigma \subseteq \Delta(\mathfrak{n})\}$ keeps stable also the complement of $\Delta(\mathfrak{n})$

$$\sigma\Delta(\mathfrak{n}) \subseteq \Delta_+ \qquad \Leftrightarrow \qquad \sigma^{-1}\Delta(\mathfrak{b}^0/\mathfrak{n}) \subseteq \Delta_+ \, .$$

The root system of $\mathfrak{m}_1$ is $\Delta(\mathfrak{m}_1) = \{e_i - e_j, i < j\}$ therefore $\omega_I \tau_I \in W^1$ implies $\tau_I^{-1}\omega_I^{-1}\Delta(\mathfrak{m}_1) \subseteq \Delta_+$ or

$$\tau_I \omega_I^{-1}(e_i - e_j) > 0 \qquad i < j \, .$$

These inequalities are satisfied for $\omega_I \in S_n$ defined by

$$\omega_I(a) > \omega_I(b) \quad \text{when} \quad \begin{cases} a < b & a \in I \quad b \in I \\ a > b & a \notin I \quad b \notin I \\ & a \in I \quad b \notin I \end{cases} \, .$$

The permutation places all elements of $I = \{i_1, \ldots i_r\}$ after all the elements of its complement $\bar{I}$ preserving the order of $\bar{I}$ and reversing the order of I, that is

$$\omega_I(1, \ldots, i_1, \ldots, i_r, \ldots, n) = (1, \ldots, \hat{i}_1, \ldots, \hat{i}_r, \ldots, n, i_r, \ldots, i_2, i_1) \, . \tag{18}$$

The permutation ω_I can be represented as a product of cyclic permutations

$$\omega_I = \zeta_{i_r} \ldots \zeta_{i_2}\zeta_{i_1}$$

where ζ_{i_k} is the place permutation cycling places from $i_k - k + 1$ to $n - k + 1$ (the cycle is of length $n - i_k + 1$). Thus the action of ω_I is represented by the sequence of steps

$$\begin{aligned} \zeta_{i_1}(1, \ldots, i_1, \ldots, i_k, \ldots n) &= (1, \ldots, \hat{i}_1, i_1 + 1, \ldots, n, i_1), \\ \zeta_{i_2}(1, \ldots, \underbrace{i_2}_{\text{place } i_2 - 1}, \ldots, n, i_1) &= (1, \ldots, \hat{i}_2, \ldots, n, i_2, i_1), \\ &\ldots \\ \zeta_{i_k}(1, \ldots, \underbrace{i_k}_{\text{place } i_k - k + 1}, \ldots, n, i_{k-1}, \ldots, i_1) &= (1, \ldots, \hat{i}_k, \ldots, n, i_k, \ldots, i_1) \, . \end{aligned}$$

Note that after the j-th step, the last j places are not touched by further cyclings.

The Weyl vector ρ associated to $\mathfrak{g} = \mathfrak{so}_{2n+1}$ reads $\rho = \frac{1}{2}\sum_{i=1}^n (2n - 2i + 1)e_i$. Note that the components of ρ are strictly decreasing with step $1 = \rho_{i+1} - \rho_i$. The highest weights $\lambda_I = \sigma(\rho) - \rho$ for $\sigma \in W^1$ are non-positive due to $\sigma(\rho)_i \leq \rho_i$. The cycling structure of ω_I implies the following form of $\lambda_I = \sum \lambda_j e_j$

$$\lambda_j = -(n - i_{n-j+1} + 1)\chi_{(n-r+1 \leq j \leq n)} - \sum_{k=1}^r \chi_{(i_k - k + 1 \leq j \leq n - k)} \, .$$

One has an isomorphism between a HW $\mathfrak{gl}_n$-module V^{λ_I} with negative weight $\lambda_I \leq 0$ and the dual representation $V^{*\mu_I}$ with reflected weight $\mu_I \geq 0$

$$V^{\lambda_I} \cong V^{*\mu_I} \qquad\qquad \mu_I := \sum_{i=1}^{n} \mu_i e_i = -\sum_{i=1}^{n} \lambda_{n-i+1} e_i \geq 0 \;.$$

The components of μ_I are decreasing positive integers $\mu_1 \geq \ldots \geq \mu_n \geq 0$

$$\mu_j = (n - i_j + 1)\chi_{(1\leq j\leq r)} + \sum_{k=1}^{r} \chi_{(k+1\leq j\leq n-i_k+k)} \;, \qquad (19)$$

and these components code a self-conjugated Young diagram $\mu'_I = \mu_I$

$$\mu_I = (\alpha_I|\alpha_I) \qquad \alpha_I = (\alpha_1, \ldots, \alpha_r), \quad \text{for} \quad \alpha_j = n - i_j \;.$$

Roughly speaking the j-th cyclic permutation ζ_{i_k} in ω_I creates a self-conjugated hook subdiagram of μ_I with $\alpha_j = \beta_j = n - i_j$.

There exist a maximal diagram $\mu_{\{1,\ldots,n\}} = \Omega_n$ corresponding to the square diagram $\Omega_n := (n^n)$. For a subset $I \subseteq \{1, \ldots, n\}$ we get self-conjugated Young diagram μ_I included into the square diagram $\mu \subseteq \Omega_n$. The empty subset gives the empty diagram $\mu_\varnothing = \varnothing$. The number of self-conjugated Young diagrams, such that $\mu_I \subseteq \Omega_n$ is 2^n so these are in bijection with elements of W^1.

By virtue of the Kostant's theorem [11] the cohomology $H^\bullet(\mathfrak{n}, \mathbb{C})$ of the free 2-step nilpotent Lie algebra $\mathfrak{n}$ has decomposition into Schur modules with HW vector $|\mu_I\rangle$

$$H^\bullet(\mathfrak{n}, \mathbb{C}) = \bigoplus_{\mu_I : \mu'_I = \mu_I} V^{*\mu_I}, \qquad |\mu_I\rangle = E^{-\Phi_\sigma}, \quad \sigma \in W^1,$$

labelled by self-conjugated Young diagrams.

The cohomological degree of the elements in $V^{*\mu_I}$ is determined the cardinality of the set $\#\Phi_\sigma$ for any $\sigma \in W^1$. Let $\sigma = \omega_I \tau_I \in W^1$. A root $\xi \in \Phi_\sigma \subseteq \Delta(\mathfrak{n})$ whenever $\sigma^{-1}\xi < 0$ in view of $\Phi_\sigma = \Delta_- \cap \sigma^{-1}\Delta_+$. But the set $\Delta(\mathfrak{n})$ is stable under permutations and $\tau_I^{-1} = \tau_I$ thus

$$\begin{aligned}
\#\Phi_\sigma &= \#\{\xi \in \Delta(\mathfrak{n}), \tau_I \xi < 0\} \\
&= \#\{\mathfrak{g}_{e_i}, i \in I\} + \#\{\mathfrak{g}_{e_i+e_j} : i < j, i \in I\} \\
&= \sum_{i\in I}(1 + n - i) = r + \sum_{k=1}^{r}(n - i_k) = r + s = \deg \mu_I \;.
\end{aligned}$$

Thus the cohomological degree $k = \deg \mu_I = \#\Phi_\sigma$ is the total degree $k = (r+s)$ of the bi-complex $\wedge^s(\wedge^2 V^*) \otimes \wedge^s V^*$. The number of boxes above the diagonal in μ_I is $s = \frac{1}{2}(|\mu_I| - r)$ so finally one gets

$$\deg \mu_I = \frac{1}{2}(r(\mu_I) + |\mu_I|) \;.$$

In particular, the degree $k_{max} = N$ of the maximal square diagram $\Omega_n = (n^n)$ is

$$N = \deg \Omega_n = n + \frac{n(n-1)}{2} = \frac{n(n+1)}{2} \ .$$

□

We got the cohomology $H^\bullet(\mathfrak{n}, \mathbb{C})$ but also the homology $H_\bullet(\mathfrak{n}, \mathbb{C})$ thanks to the isomorphism (15). The homology $H_\bullet(\mathfrak{n}, \mathbb{C})$ was first calculated by Jósefiak and Weyman [7] and then independently using different method by Sigg [18].

3.5. The cohomology $H^\bullet(\mathfrak{n}, \mathcal{V}(p))$

The application of the Kostant theorem [11] for calculation of $H^\bullet(\mathfrak{n}, \mathbb{C})$ [5] has the advantage to be generalizable for cohomologies with coefficients in the parafermionic Fock space $\mathcal{V}(p)$.

Corollary 3.6. *Let $\mathfrak{n}$ be the free 2-step nilpotent Lie algebra $\mathfrak{n} = V \oplus \wedge^2 V$ and $V^\Lambda = \mathcal{V}(p)$ be the parafermionic Fock space. The cohomology $H^\bullet(\mathfrak{n}, V^\Lambda)$ with values in the $V^\Lambda \downarrow^{\mathfrak{g}}_{\mathfrak{n}}$ has the following decomposition into irreducible $\mathfrak{gl}(V)$-modules*

$$H^k(\mathfrak{n}, V^\Lambda) \cong \bigoplus_{\mu : \mu = \mu'} V^{* \mu^{(p)}} \ , \qquad\qquad k = \frac{1}{2}(|\mu| + r(\mu)) \ ,$$

where the sum is over self-conjugated Young diagrams $\mu = (\alpha|\alpha)$ and the notation $\mu^{(p)}$ stays for the p-augmented diagram $\mu^{(p)} = (\alpha + p|\alpha)$.

Note that the cohomology $H^\bullet(\mathfrak{n}, \mathbb{C})$ corresponds to the case $p = 0$. The result follows again from the Kostant's theorem, but now the HW $\mathfrak{gl}(V)$-modules are built on the HW weights $\sigma(\rho + \Lambda) - \rho$ where $\Lambda = \frac{p}{2} \sum e_i$. The permutation $\sigma = \omega_I \tau_I \in W^1$ is again given by eq. (18). The shift Λ modifies the dominant weight $\nu_I = \sum \nu_i e_i$ for given $I = \{i_1, \ldots, i_r\}$ as follows

$$\nu_j = \frac{p}{2} + (n - i_j + 1 + p)\chi_{(1 \leq j \leq r)} + \sum_{k=1}^{r} \chi_{(k+1 \leq j \leq n - i_k + k)} \ .$$

But these are the HW $\mathfrak{gl}(V)$-weights of the HW vectors of $\mathfrak{gl}(V)$-modules

$$V^{* \mu^{(p)}} \otimes_{\mathfrak{n}^*} |\Lambda\rangle$$

where $\mu^{(p)}$ is one of the diagrams $\mu_I^{(p)} = (\alpha|\beta)$ with arm-lengths $\alpha_j(\mu_I^{(p)}) = n - i_j + p$ and leg-lengths $\beta_j(\mu_I^{(p)}) = n - i_j$, for $j = 1, \ldots, r$. The arm p-augmented diagram $\mu^{(p)}$ results from the symmetric diagram $\mu = (\beta|\beta)$ (19) by augmenting the arm-lengths (see eq. (8)). The sum over $\sigma \in W^1$ in Kostant's theorem is equivalent to the sum over self-conjugated Young diagrams $\mu = \mu'$. □

4. Resolution of $\mathcal{V}(p)$

According to a general result of Henri Cartan [1] every graded $\mathcal{A}$-module M of a graded algebra $\mathcal{A}$ allows for a minimal projective resolution by projective $\mathcal{A}$-modules and moreover the notions of a projective and a free module coincide in the graded category. Thus for every graded $\mathcal{A}$-module M we end up with a free resolution.

The universal enveloping algebra $U\mathfrak{n}$ is a graded associative algebra and the parafermionic Fock space $\mathcal{V}(p) = V^{\Lambda}$ is a graded $U\mathfrak{n}$-module. Therefore there exists a minimal free resolution $P_\bullet = \bigoplus_{k=0}^{N} P_k$ of the right $U\mathfrak{n}$-module $\mathcal{V}(p)^*$

$$0 \to P_N \to \ldots \to P_1 \to P_0 \stackrel{\epsilon}{\to} \mathcal{V}(p)^* \to 0 \tag{20}$$

by some free right $U\mathfrak{n}$-modules $P_k = E_k \otimes U\mathfrak{n}$. Here the map ϵ is the augmentation of $U\mathfrak{n}$. The minimality of the resolution implies that the differentials of the derived complex $P_\bullet \otimes_{U\mathfrak{n}} \mathbb{C}$ vanish [1]. Hence the multiplicity spaces E_k coincide with the homologies of $\mathfrak{n}$ with coefficients in the right module $\mathcal{V}(p)^*$

$$E_k \cong \mathrm{Tor}_k^{U\mathfrak{n}}(\mathcal{V}(p)^*, \mathbb{C}) = H_k(\mathfrak{n}, \mathcal{V}(p)^*) \ .$$

But as the homology and cohomology of $\mathfrak{n}$ are related by an isomorphism (14) we find the vector space isomorphism

$$E_k^* \cong H^k(\mathfrak{n}, \mathcal{V}(p)) \ .$$

hence by Corollary 3.6. we have constructed the minimal free resolution (20).

Proposition 4.1. *The Euler-Poincare characteristic of the free minimal resolution of the (dual of the) parafermionic Fock space $\mathcal{V}(p)$ (20) yields the parafermionic sign-alternating identity (7) found in the work [19] of Stoilova and Van der Jeugt.*

Proof. In general, the mapping of modules of an algebra into its Grothendieck ring of characters is an exemple of Poincare-Euler characteristic. The free resolution (20) is naturally a (reducible) $\mathfrak{gl}(V)$-module and the Schur functions (5) span the ring of $\mathfrak{gl}(V)$-characters. All the homology of a resolution is concentrated in degree 0, hence on the RHS of (7) stays the character of $\mathcal{V}(p)$(6)

$$char\mathcal{V}(p) = e^{p\theta} \sum_{\lambda \subseteq (p^n)} s_\lambda(x) \ .$$

From the Poincaré-Birkhof-Witt theorem follows that the character of $U\mathfrak{n}$ (see [2, 3]) is

$$charU\mathfrak{n} = \frac{1}{\prod_i (1 - x_i) \prod_{i<j} (1 - x_i x_j)}$$

Thus the alternating sum on the LHS comes from the characters of the $\mathfrak{gl}(V)$-modules $E_k \otimes U\mathfrak{n}$ taken with alternating signs corresponding to the homological degree. The factor $e^{p\theta}$ which is the weight of the HW vector $|\Lambda\rangle$ cancels which proves the parafermionic sign-alternating identity (7). $\square$

Remark. When $p = 0$ the free minimal resolution of the trivial module $\mathbb{C}$(20) with multiplicity spaces E_k associated to the homologies $H_k(\mathfrak{n}, C)$ was constructed by Józefiak and Weyman [7]. Thus the sign-alternating character identity (7) with $p = 0$ gives back a homological explanation of the "Littlewood formula"[7]

$$\frac{\sum_{\eta:\eta'=\eta}(-1)^{(|\eta|+r)/2}s_\eta(x)}{\prod_i(1-x_i)\prod_{i<j}(1-x_ix_j)} = 1\ . \tag{21}$$

5. Kadeishvili Homotopy Transfer Theorem

Let A be the complex of morphisms of the free resolution $P_\bullet$ (20) of the right $U\mathfrak{n}$-module $\mathcal{V}(p)^*$

$$A = \mathrm{Hom}_{U\mathfrak{n}}(P_\bullet, P_\bullet) = \mathrm{End}_{U\mathfrak{n}}P_\bullet\ .$$

The graded morphisms on A are the graded $U\mathfrak{n}$-linear maps $f : P_\bullet \to P_\bullet$. The differential in A is the the differential of the map

$$d(f) = d_P \circ f - (-1)^{\deg f} f \circ d_P\ .$$

Therefore A is a differential graded algebra (DGA) of endomorphism of $P_\bullet$.

We are now in position to apply the Kadeishvili's homotopy transfer theorem [8]. It induces on the cohomology $H^\bullet A$ of the DGA A a structure of higher multiplications $m_i : A^{\otimes i} \to A$ satisfying the Stasheff coherent relations. This structure is called *homotopy associative algebra*, or A_∞-algebra for short. Whenever A is a commutative DGA there exists the refined version of *commutative* A_∞-algebra called homotopy commutative (and associative) algebra, or C_∞-algebra for short. For a friendly introduction to homotopy transfer theorems we send the reader to the lectures [9] and the textbook [14](see chapter 9).

Theorem 5.1. (Kadeishvili [8]) *Let $(A, d, \wedge)$ be a (commutative) DGA. There exists a A_∞-algebra (C_∞-algebra) structure on the cohomology $H^\bullet(A)$ and a $A_\infty(C_\infty)$-quasi-isomorphism $f_i : (\otimes^i H^\bullet(A), \{m_i\}) \to (A, \{d, \wedge, 0, 0, \ldots\})$ such that the inclusion $f_1 = i : H^\bullet(A) \to A$ is a cocycle-choosing homomorphism of cochain complexes. The differential m_1 on $H^\bullet(A)$ is zero ($m_1 = 0$) and m_2 is strictly associative operation induced by the multiplication on A. The resulting structure is unique up to quasi-isomorphism.*

By virtue of the Kadeishvili homotopy transfer theorem [8] the homology algebra $H^\bullet A$ of the DGA A is a A_∞-algebra. The cohomology $H^\bullet A$ of the DGA $A = \mathrm{Hom}_{U\mathfrak{n}}(P_\bullet, P_\bullet)$ is isomorphic as m_2-algebra to the extension algebra of $\mathcal{V}(p)^*$

$$H^\bullet A = H^\bullet(\mathrm{Hom}_{U\mathfrak{n}}(P_\bullet, P_\bullet)) \cong \mathrm{Ext}^\bullet_{U\mathfrak{n}}(\mathcal{V}(p)^*, \mathcal{V}(p)^*)\ .$$

Hence the algebra $\mathrm{Ext}^\bullet_{U\mathfrak{n}}(\mathcal{V}(p)^*, \mathcal{V}(p)^*)$ is endowed with a structure of A_∞.

The study of the latter A_∞-algebra is a challenging problem involving representation theory and combinatorics of Young tableaux. We have made already some progress in dealing with the higher structure in the particular case $p = 0$, that is, on the cohomology of the 2-step nilpotent free Lie algebra

$$H^\bullet(\mathfrak{n}, \mathbb{C}) = \mathrm{Ext}^\bullet_{U\mathfrak{n}}(\mathbb{C}, \mathbb{C}) .$$

Due to the (super-)commutativity of the cohomology ring $H^\bullet(\mathfrak{n}, \mathbb{C})$ the induced A_∞-algebra enjoys additional symmetries and provides an example of *commutative* (and associative) homotopy algebra. For a report on the C_∞-structure on $H^\bullet(\mathfrak{n}, \mathbb{C})$ see the contribution to the XXVII Geometrical Seminar in Zlatibor, 2012 [4].

6. Acknowledgements

It is my pleasure to thank Michel Dubois-Violette for reading the manuscript and for many enlightening discussions. I'd also like to thank the Serbian hosts for the warm hospitality and the financial support during the 7th Mathematical Physics Meeting at Belgrade.

References

[1] H. Cartan. *Homologie et cohomologie d' une algèbre graduée. Séminaire Henri Cartan*, **11**(1958), 1-20.

[2] S. Chaturvedi, *Canonical partition functions for parastatistical systems of any order Phys. Rev. E* **54** (1996), 1378-1382.

[3] M. Dubois-Violette, T. Popov, *Homogeneous Algebras, Statistics and Combinatorics, LMP* **61**(2002), 159-170.

[4] M. Dubois-Violette, T. Popov, *2-Step Nilpotent Lie algebras and C_∞ - algebras. Proceedings of XVII Geometrical Seminar*, Zlatibor 2012 ed. Zoran Rakic (to appear).

[5] J. Grassberger, A. King and P. Tirao. *On the homology of free 2-step nilpotent Lie algebras. Journal of Algebra* **254**(2002), 213-225.

[6] H. S. Green, *A Generalized Method of Field Quantization*, *Phys. Rev.* **90**(1953), 270-273.

[7] T. Józefiak, J. Weyman, *Representation-theoretic interpretation of a formula of D. E. Littlewood,Math. Proc. Cambridge Phil. Soc.*, **103**(1988), 193-196.

[8] T. Kadeishvili, *The A_∞-algebra Structure and Cohomology of Hochschild and Harrison*, *Proc. of Tbil. Math. Inst.* **91**(1988), 19-27.

[9] B. Keller, *Introduction to A_∞-algebras and modules. Homology, Homotopy and Applications* **3**(2001),1-35.

[10] R. King, *From Palevs Study of Wigner Quantum Systems to New Results on Sums of Schur Functions*, *in Proceedings of the IX International Workshop "Lie Theory and Its Applications in Physics"*, Vl. Dobrev eds, Springer Proceedings in Mathematics (2012).

[11] Kostant, Bertram. *Lie Algebra Cohomology and the Generalized Borel-Weil Theorem. The Annals of Mathematics* **74** (1961), 329-387.

[12] J.-L. Loday and T. Popov, Algebras of Young Tableaux and Parastatistics. *Proceedings of the VII International Workshop "Lie Theory and Its Applications in Physics"*, eds. H.-D. Doebner and V.K. Dobrev, (Heron Press, Sofia, 2008), 423-430.

[13] J.-L. Loday and T. Popov, *Parastatistics Algebra, Young Tableaux and the Super Plactic Monoid. Int. J. Geom. Meth. in Mod. Phys.* **5** (2008), 1295-1314.

[14] J.-L. Loday, B. Vallette, *Algebraic Operads*, Grundlehren Math. Wiss. **346**, Springer, Heidelberg, 2012.

[15] S. Lievens, N. I. Stoilov, and J. Van der Jeugt, *The Paraboson Fock Space and Unitary Irreducible Representations of the Lie Superalgebra. Commun. Math. Phys.* **281** (2008), 805-826.

[16] I.G. Macdonald, *Symmetric functions and Hall polynomials*, Oxford, 1979.

[17] T. Palev, Para-Bose and para-Fermi operators as generators of orthosymplectic Lie superalgebras, *Journal of Math. Phys.* **23** (1982), 1100-1102.

[18] S. Sigg, *Laplacian and homology of free 2-step nilpotent Lie algebras*, J. Algebra **185**(1996), 144-161.

[19] N.I. Stoilova and J. Van der Jeugt, *The parafermion Fock space and explicit* $\mathfrak{so}(2n+1)$ *representations, J. Phys. A: Math. Theor.* **41** (2008), 075202.

[20] N. I. Stoilova and J. Van der Jeugt, Parafermions, parabosons and representations of so(∞) and osp(1|∞) *International Journal of Mathematics* **20** (2009), 693–715.