Matrix superpotentials of the special form

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Abstract

Matrix superpotentials that correspond to exactly solvable Scrödinger equations are classified. We consider superpotentials of the special form $W_k = kQ + \frac{1}{k}R$, where k is a parameter, P and R are Hermitian matrices depending on the variable x. A classification list of the two-dimensional matrix superpotentials is presented.

1. Introduction

Supersymmetry represents an efficient tool for solving quantum mechanical problems, described by Schrödinger equations. Unfortunately a class of known problems that can be solved due to their supersymmetric nature is rather restricted, because they should satisfy an additional condition called shape-invariance [2]. The classification of scalar shape-invariant superpotentials was considered in [3]. Matrix superpotentials mostly appear as important but stand alone examples. The well-known example of such a problem is the Pron'ko-Stroganov problem [4], which supersymmetric nature was described in [5]. Another examples of supersymmetric matrix problems can be found in [6].

A systematic study of matrix superpotentials has been presented in recent papers [**?**, 8], that consider superpotentials of the form $W_k = kQ + P + \frac{1}{k}R$, where k is a parameter, P, Q and R are Hermitian matrices that satisfy one of the following conditions: either the matrix Q is proportional to the unit matrix, or the matrix R is the zero matrix. In paper [9] a wider class of superpotentials was described, but the classification was not completed.

In this paper the important case $W_k = kQ + \frac{1}{k}R$ is investigated. This case completes the class of superpotentials considered in previous papers. A list of the two-dimensional matrix superpotentials is presented explicitly.

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2. Shape-invariant spectral problems

We consider the spectral problem

$$H_k \psi = E_k \psi, \tag{1}$$

where H_k is a Hamiltonian with matrix superpotential, E_k and ψ are its eigenvalues and eigenfunctions correspondingly. We will look for solutions ψ_i in the class of square-integrable functions, which smoothly tend to zero at bounds of considering interval.

Assume that the Hamiltonian can be represented in the following form

$$H_k = a_k^\dagger a_k + c_k, \tag{2}$$

where c_k is a scalar function of k, which vanishes with the corresponding member of Hamiltonian.

It is possible to restrict ourselves (see [8] for proof) by operators a_k and $a_k^\dagger$ of the form

$$a_k = \frac{\partial}{\partial x} + W_k(x), \quad a_k^\dagger = -\frac{\partial}{\partial x} + W_k(x), \tag{3}$$

where W_k is a Hermitian matrix called superpotential.

Additionally, if the spectral problem satisfies the condition

$$H_k^+ = H_{k+1}, \tag{4}$$

where H_k^+ is a superpartner of the Hamiltonian defining by the formula

$$H_k^+ = a_k a_k^\dagger + c_k, \tag{5}$$

then it is called a shape-invariant spectral problem and can be solved algebraically. A method of the solution of shape-invariant problems can be found e.g. in [3].

3. Classification problem

Let us formulate the problem to find all matrix superpotentials, that correspond to shape-invariant matrix problems and have the following form

$$W_k = k\,Q + \frac{1}{k} R, \tag{6}$$

where matrices Q and R are not special (not equal to zero or proportional to unit matrix).

Substituting the expression of the superpotential into equations (2), (4) and splitting the obtained equation with respect to the variable x and the parameter k, we get the system of determining equations for unknown matrices Q and R:

$$Q' = Q^2 + \nu, \tag{7}$$

$$R' = 0, \tag{8}$$

$$R^2 = \omega^2, \tag{9}$$

$$\delta_k = (2k+1)\nu + \frac{(2k+1)\omega^2}{k^2(k+1)^2}. \tag{10}$$

We can see that the system consists of independent equations (7) for the matrix Q, (8) and (9) for the matrix R and equation (10) for defining δ_k.

The equation (7) is equal to the one considered in the article [8], and can be solve in the same manner. Hence matrix Q can be reduced to the diagonal form

$$Q = \mathrm{diag}\{q_1, \ldots, q_n\} \tag{11}$$

by means of unitary transformation which does not depend on variable x, viz equivalence transformation.

Equations (8) and (9) for unknown matrix R were considered in [7]. It was shown there that the matrix R is a constant matrix with square proportional to the unit one. In the same paper it was noted that an equivalence transformation can reduce matrix R to the following form

$$\tilde{R} = \omega \begin{pmatrix} I_{m\times m} & 0_{m\times s} \\ 0_{s\times m} & -I_{s\times s} \end{pmatrix}, \quad m + s = n, \tag{12}$$

where I and 0 are unit and zero matrices correspondingly, with dimension indicated by subindexes.

Therefore independently simplifying matrices Q and R with equivalence transformation we can reduce the matrices to forms (11) and (12) correspondingly. But when we simplify the matrix Q we change the matrix R at the same time and vice versa, thus simultaneous transformations are not possible.

Assume that we simplify the matrix R with some equivalence transformation and reduce it to the form (12), at this time the matrix Q has the most general form. Assume also that the unitary matrix U corresponds to the equivalence transformation, that diagonalizes the matrix Q. Therefore superpotential of the form (6) can be described by matrix Q of the form (11) and by the matrix R of the form

$$R = U\tilde{R}U^{\dagger}, \tag{13}$$

where the matrix $\tilde{R}$ is defined by the formula (12).

Note that since we multiply the matrix R by the matrix Q from both sides we can restrict U to be special unitary matrix instead of the unitary one (it can be defined as $U/\det U$). We are looking for irreducible matrix superpotentials so we should consider only matrices U that are not block-diagonal, otherwise the corresponding superpotential is completely reducible.

In this way we reduce the classification problem of shape-invariant superpotentials (6) to the problem of solving the system of ordinary differential equations of the first order, i.e. Riccati equations for functions q_i; and enumeration of all special unitary matrices U, that are not block diagonal.

4. Superpotentials and shape-invariant potentials

In this section we finally describe superpotentials of the form (6). The matrix Q is defined by its diagonal elements q_i, that can be found from equation (7), that is easily integrated and have the following solutions

$$
\begin{aligned}
&q_i = \left[\begin{array}{ll} -\dfrac{1}{x+\gamma_i}, & i=1,\dots,m, \\ 0, & i=m+1,\dots,n, \end{array}\right. \qquad \nu = 0; \\
&q_i = \lambda\tan(\lambda x+\gamma_i), \quad i=1,\dots,n, \quad \nu=\lambda^2>0; \\
&q_i = \left[\begin{array}{ll} -\lambda\tanh(\lambda x+\gamma_i), & i=1,\dots,m, \\ -\lambda\coth(\lambda x+\gamma_i), & i=m+1,\dots,l, \\ \pm\lambda, & i=l+1,\dots,n, \end{array}\right. \qquad \nu=-\lambda^2<0,
\end{aligned}
\tag{14}
$$

where $\gamma_i \in \mathbb{R}$, $i=1,\dots,n$, are integration constants.

In cases when $\nu<0$ and $\nu=0$ the matrix Q consists of blocks of dimension $m, l-m+1, n-l+1$ and $m, n-m+1$ correspondingly. Some of this blocks may or may not appear, namely have zero size. In case when the matrix Q is a constant matrix the potential is also a constant matrix, so this case is not important.

As it was mentioned before the matrix R is defined by special unitary transformation which is not block-diagonal.

Thus we obtained the most general presentation of the matrices Q by formula (14), and for the matrices R by formulas (13) and (12). Corresponding potentials can be obtained using the formula

$$V_k = -W_k' + W_k^2 + c_k. \tag{15}$$

Using the obtained results we write down the list of two-dimensional superpotentials of the form (6). Note that for arbitrary dimension n the set of all superpotentials of the form (6) is equipotent to the set of all non block-diagonal unitary matrices of dimension n. Therefore for arbitrary not fixed n all superpotentials can't be presented explicitly, inspite of the fact that for any fixed n this procedure is rather simple.

Let us agree that in case we have two integration constants γ_1 and γ_2 we shift variable x in the way $\gamma_1=\gamma$ and $\gamma_2=-\gamma$, and in the case we have only one integration constant — $\gamma_1 = 0$. It is possible as well to make all elements of the matrix R real, so that it has the form

$$R = \begin{pmatrix} \rho & \varepsilon \\ \varepsilon & -\rho \end{pmatrix} = \varepsilon\,\sigma^1 + \rho\,\sigma^3,$$

where $\rho^2 + \varepsilon^2 = \omega^2$ and $\varepsilon \neq 0$.

For convenience we use matrices σ^+ and σ^-, defined by the formulas

$$\sigma^+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \frac{1}{2}\,(\sigma^0 + \sigma^3), \quad \sigma^- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{2}\,(\sigma^0 - \sigma^3).$$

As a result we obtain eight new two-dimensional superpotentials

$$W_k = \left(\frac{\rho}{k} - \frac{k}{x+\gamma}\right)\sigma^+ - \left(\frac{\rho}{k} + \frac{k}{x-\gamma}\right)\sigma^- + \frac{\varepsilon\sigma^1}{k},$$

$$W_k = \left(\frac{\rho}{k} - \frac{k}{x}\right)\sigma^+ - \frac{\rho\sigma^-}{k} + \frac{\varepsilon\sigma^1}{k},$$

$$W_k = \left(\frac{\rho}{k} + k\lambda\tan(\lambda x+\gamma)\right)\sigma^+ + \frac{\varepsilon\sigma^1}{k} - \left(\frac{\rho}{k} - k\lambda\tan(\lambda x-\gamma)\right)\sigma^-,$$

$$W_k = \left(\frac{\rho}{k} - k\lambda\tanh(\lambda x+\gamma)\right)\sigma^+ + \frac{\varepsilon\sigma^1}{k} - \left(\frac{\rho}{k} + k\lambda\tanh(\lambda x-\gamma)\right)\sigma^-,$$

$$W_k = \left(\frac{\rho}{k} - k\lambda\coth(\lambda x+\gamma)\right)\sigma^+ + \frac{\varepsilon\sigma^1}{k} - \left(\frac{\rho}{k} + k\lambda\coth(\lambda x-\gamma)\right)\sigma^-,$$

$$W_k = \left(\frac{\rho}{k} - k\lambda\tanh(\lambda x+\gamma)\right)\sigma^+ + \frac{\varepsilon\sigma^1}{k} - \left(\frac{\rho}{k} + k\lambda\coth(\lambda x-\gamma)\right)\sigma^-,$$

$$W_k = \left(\frac{\rho}{k} - k\lambda\tanh\lambda x\right)\sigma^+ - \left(\frac{\rho}{k} \pm k\lambda\right)\sigma^- + \frac{\varepsilon\sigma^1}{k},$$

$$W_k = \left(\frac{\rho}{k} - k\lambda\coth\lambda x\right)\sigma^+ - \left(\frac{\rho}{k} \pm k\lambda\right)\sigma^- + \frac{\varepsilon\sigma^1}{k}.$$

Corresponding shape-invariant matrix potentials can be easily constructed using (15).

5. Conclusion

In the present paper superpotentials of the form $W_k = kQ + \frac{1}{k}R$ were considered and all corresponding shape-invariant potentials were found. The list of two-dimensional superpotentials was presented explicitly. All eight superpotentials that were found are new. The problem of classification of superpotentials of arbitrary order n was reduced to the problem of enumeration of the non block-diagonal special unitary matrices.

The obtained superpotentials correspond to systems of Schrödinger equations and can be used to solve the later. Though the class of found superpotentials is infinitely wide it does not exhaust all shape-invariant matrix problems. In the future works the classification will be continued and a wider class will be considered.

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