Special Reduced Multiplets and Minimal Representations for Sp(n,R)

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Abstract

In the present paper we continue the programme of systematic construction of invariant differential operators on the example of the non-compact groups Sp(n,R). Earlier in arXiv:1205.5521 we gave the main multiplets and the main reduced multiplets of indecomposable elementary representations including the necessary data for all relevant invariant differential operators. Here we give the special reduced multiplets and the minimal representations of Sp(n,R).

1. Introduction

Invariant differential operators play very important role in the description of physical symmetries. For the modern applications of (super-)differential operators in conformal field theory, supergravity and string theory we refer, e.g., to [1].

In a recent paper [2] we started the systematic explicit construction of invariant differential operators. We gave an explicit description of the building blocks, namely, the parabolic subgroups and subalgebras from which the necessary representations are induced. Thus we have set the stage for study of different non-compact groups.

In the present paper we focus on the groups $Sp(n, I\!R)$, which are very interesting for several reasons. First of all, they belong to the class of Hermitian symmetric spaces, i.e., the pair (G, K) is a Hermitian symmetric pair (K is the maximal compact subgroup of the noncompact semisimple group G). These groups have discrete series representations and highest/lowest weight representations. Further, $Sp(n, I\!R)$ belong to a narrower class of groups/algebras, which we call 'conformal Lie groups or algebras' since they have very similar properties to the canonical conformal algebras $so(n, 2)$ of n-dimensional Minkowski space-time. This class was identified from our point of view in [3]. Besides $so(n, 2)$ it includes the algebras $su(n, n)$,

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$sp(n, I\!R)$, $so^*(4n)$, $E_{7(-25)}$. The same class was identified independently from different considerations and under different names in [4, 5, 6].

This paper is a sequel of [7], based on Invited talk at the VII Mathematical Physics Meeting, Belgrade, 9-19.9.2012. Due to the lack of space we refer to [7] for motivations and extensive list of literature on the subject.

The present paper is organized a follows. In section 2 we give the preliminaries, actually recalling and adapting facts from [2] to the $sp(n, I\!R)$ case. In Section 3 we present the special reduced multiplets $n = 2, ..., 6$ and the intertwining differential operators between the ERs. In Section 4 we discuss the general features of our results which generalize for arbitrary n. We also present an Outlook.

2. Preliminaries

Let $n \geq 2$. Let $\mathcal{G} = sp(n, I\!R)$, the split real form of $sp(n, \mathbb{C}) = \mathcal{G}^{\mathbb{C}}$. The maximal compact subgroup of $\mathcal{G}$ is $\mathcal{K} \cong u(1) \oplus su(n)$.

We choose a *maximal* parabolic $\mathcal{P} = \mathcal{M}\mathcal{A}\mathcal{N}$ such that $\mathcal{A} \cong so(1,1)$, while the factor $\mathcal{M} = sl(n, I\!R)$ has the same finite-dimensional (nonunitary) representations as the finite-dimensional (unitary) representations of the semi-simple subalgebra $su(n)$ of $\mathcal{K}$. Thus, these induced representations are representations of finite $\mathcal{K}$-type [8]. Note also that $\mathcal{K}^{\mathbb{C}} \cong u(1)^{\mathbb{C}} \oplus sl(n, \mathbb{C}) \cong \mathcal{M}^{\mathbb{C}} \oplus \mathcal{A}^{\mathbb{C}}$. Finally, note that $\dim_{I\!R} \mathcal{N} = n(n+1)/2$.

We label the signature of the ERs of $\mathcal{G}$ as follows:

$$\chi = \{ n_1, \ldots, n_{n-1} ; c \} , \qquad n_j \in I\!N , \quad c = d - (n+1)/2 \qquad (1)$$

where the last entry of χ labels the characters of $\mathcal{A}$, and the first $n-1$ entries are labels of the finite-dimensional nonunitary irreps of $\mathcal{M}$, (or of the finite-dimensional unitary irreps of $su(n)$).

Below we shall use the following conjugation on the finite-dimensional entries of the signature:

$$(n_1, \ldots, n_{n-1})^* \doteq (n_{n-1}, \ldots, n_1) \qquad (2)$$

We call the above induced representations $\chi = \mathrm{Ind}_{\mathcal{P}}^{G}(\mu \otimes \nu \otimes 1)$ *elementary representations* [9] of $G = Sp(n, I\!R)$. (These are called *generalized principal series representations* (or *limits thereof*) in [10].) Their spaces of functions are:

$$\mathcal{C}_\chi = \{\mathcal{F} \in C^\infty(G, V_\mu) \mid \mathcal{F}(g\hat{m}\hat{a}\hat{n}) = e^{-\nu(H)} \cdot D^\mu(\hat{m}^{-1})\, \mathcal{F}(g)\}$$

where $\hat{a} = \exp(H)$, $H \in \mathcal{A}$, $\hat{m} \in M = SL(n, I\!R)$, $\hat{n} \in N = \exp \mathcal{N}$. The representation action is the *left regular action*:

$$(\mathcal{T}^\chi(g)\mathcal{F})(g') = \mathcal{F}(g^{-1}g') , \quad g, g' \in G . \qquad (3)$$

• An important ingredient in our considerations are the *highest/lowest weight representations* of $\mathcal{G}^{\mathbb{C}}$. These can be realized as (factor-modules of) Verma modules V^Λ over $\mathcal{G}^{\mathbb{C}}$, where $\Lambda \in (\mathcal{H}^{\mathbb{C}})^*$, $\mathcal{H}^{\mathbb{C}}$ is a Cartan subalgebra of $\mathcal{G}^{\mathbb{C}}$, weight $\Lambda = \Lambda(\chi)$ is determined uniquely from χ [11].

Actually, since our ERs are induced from finite-dimensional representations of $\mathcal{M}$ the Verma modules are always reducible. Thus, it is more convenient to use *generalized Verma modules* $\tilde{V}^\Lambda$ such that the role of the highest/lowest weight vector v_0 is taken by the (finite-dimensional) space $V_\mu \, v_0$. For the generalized Verma modules (GVMs) the reducibility is controlled only by the value of the conformal weight d, or the parameter c. Relatedly, for the intertwining differential operators only the reducibility w.r.t. non-compact roots is essential.

• Another main ingredient of our approach is as follows. We group the (reducible) ERs with the same Casimirs in sets called *multiplets* [12]. The multiplet corresponding to fixed values of the Casimirs may be depicted as a connected graph, the *vertices* of which correspond to the reducible ERs and the *lines (arrows)* between the vertices correspond to intertwining operators. The multiplets contain explicitly all the data necessary to construct the intertwining differential operators. Actually, the data for each intertwining differential operator consists of the pair (β, m), where β is a (non-compact) positive root of $\mathcal{G}^{\mathbb{C}}$, $m \in \mathbb{N}$, such that the *BGG Verma module reducibility condition* [13] (for highest weight modules) is fulfilled:

$$(\Lambda + \rho, \beta^\vee) \ = \ m \ , \quad \beta^\vee \equiv 2\beta/(\beta,\beta) \tag{4}$$

where ρ is half the sum of the positive roots of $\mathcal{G}^{\mathbb{C}}$. When the above holds then the Verma module with shifted weight $V^{\Lambda - m\beta}$ (or $\tilde{V}^{\Lambda - m\beta}$ for GVM and β non-compact) is embedded in the Verma module V^Λ (or $\tilde{V}^\Lambda$). This embedding is realized by a singular vector v_s expressed by a polynomial $\mathcal{P}_{m,\beta}(\mathcal{G}^-)$ in the universal enveloping algebra $(U(\mathcal{G}_-))\ v_0$, $\mathcal{G}^-$ is the subalgebra of $\mathcal{G}^{\mathbb{C}}$ generated by the negative root generators [14]. More explicitly, [11], $v^s_{m,\beta} = \mathcal{P}_{m,\beta}\, v_0$ (or $v^s_{m,\beta} = \mathcal{P}_{m,\beta}\, V_\mu\, v_0$ for GVMs).
Then there exists [11] an *intertwining differential operator* of order $m = m_\beta$:

$$\mathcal{D}_{m,\beta} \ : \ \mathcal{C}_{\chi(\Lambda)} \ \longrightarrow \ \mathcal{C}_{\chi(\Lambda - m\beta)} \tag{5}$$

given explicitly by:

$$\mathcal{D}_{m,\beta} \ = \ \mathcal{P}_{m,\beta}(\widehat{\mathcal{G}^-}) \tag{6}$$

where $\widehat{\mathcal{G}^-}$ denotes the *right action* on the functions $\mathcal{F}$.

Thus, in each such situation we have an *invariant differential equation* of order $m = m_\beta$:

$$\mathcal{D}_{m,\beta}\ f \ = \ f' \ , \qquad f \in \mathcal{C}_{\chi(\Lambda)} \ , \quad f' \in \mathcal{C}_{\chi(\Lambda - m\beta)} \ . \tag{7}$$

In most of these situations the invariant operator $\mathcal{D}_{m,\beta}$ has a non-trivial invariant kernel in which a subrepresentation of $\mathcal{G}$ is realized. Thus, studying the equations with trivial RHS:

$$\mathcal{D}_{m,\beta}\ f \ =\ 0\ , \qquad f \in \mathcal{C}_{\chi(\Lambda)}\ , \tag{8}$$

is also very important. For example, in many physical applications in the case of first order differential operators, i.e., for $m = m_\beta = 1$, equations (8) are called *conservation laws*, and the elements $f \in \ker \mathcal{D}_{m,\beta}$ are called *conserved currents*.

The ERs in the multiplet are related also by intertwining integral operators. The integral operators were introduced by Knapp and Stein [15]. In fact, these operators are defined for any ER, not only for the reducible ones, the general action being:

$$\begin{gathered} G_{KS}\ :\ \mathcal{C}_\chi \ \longrightarrow\ \mathcal{C}_{\chi'}\ , \\ \chi \ =\ \{\, n_1, \ldots, n_{n-1}\,;\, c\,\}\ , \qquad \chi' \ =\ \{\,(n_1, \ldots, n_{n-1})^*\,;\, -c\,\} \end{gathered} \tag{9}$$

The above action on the signatures is also called restricted Weyl reflection, since it represents the nontrivial element of the 2-element restricted Weyl group which arises canonically with every maximal parabolic subalgebra. Generically, the Knapp-Stein operators can be normalized so that indeed $G_{KS} \circ G_{KS} = \mathrm{Id}_{\mathcal{C}_\chi}$. However, this usually fails exactly for the reducible ERs that form the multiplets, cf., e.g., [9].

Further, we need more explicitly the root system of the algebra $sp(n, F)$, $F = \mathbb{C}, \mathbb{R}$. In terms of the orthonormal basis ϵ_i, $i = 1, \ldots, n$, the positive roots are given by

$$\Delta^+ = \{\epsilon_i \pm \epsilon_j,\ 1 \le i < j \le n;\ 2\epsilon_i, 1 \le i \le n\}, \tag{10}$$

while the simple roots are:

$$\pi = \{\alpha_i = \epsilon_i - \epsilon_{i+1},\ 1 \le i \le n-1;\ \alpha_n = 2\epsilon_n\} \tag{11}$$

With our choice of normalization of the long roots $2\epsilon_k$ have length 4, while the short roots $\epsilon_i \pm \epsilon_j$ have length 2.

From these the compact roots are those that form (by restriction) the root system of the semisimple part of $\mathcal{K}^{\mathbb{C}}$, the rest are noncompact, i.e.,

$$\begin{aligned} \text{compact}: \qquad & \alpha_{ij} \ \equiv\ \epsilon_i - \epsilon_j\ , \quad 1 \le i < j \le n\ , \\ \text{noncompact}: \qquad & \beta_{ij} \ \equiv\ \epsilon_i + \epsilon_j\ , \quad 1 \le i \le j \le n \end{aligned} \tag{12}$$

Thus, the only non-compact simple root is $\alpha_n = \beta_{nn}$.

Further, we shall use the so-called Dynkin labels:

$$m_i \ \equiv\ (\Lambda + \rho, \alpha_i^\vee)\ , \quad i = 1, \ldots, n, \tag{13}$$

where $\Lambda = \Lambda(\chi)$, ρ is half the sum of the positive roots of $\mathcal{G}^{\mathbb{C}}$.

We shall use also the so-called Harish-Chandra parameters:

$$m_\beta \equiv (\Lambda + \rho, \beta) \ , \tag{14}$$

where β is any positive root of $\mathcal{G}^{\mathbb{C}}$. These parameters are redundant, since they are expressed in terms of the Dynkin labels, however, some statements are best formulated in their terms. In particular, in the case of the noncompact roots we have:

$$m_{\beta_{ij}} \ = \ \Big(\sum_{s=i}^{n} + \sum_{s=j}^{n}\Big) m_s \ , \quad i < j \ ; \qquad m_{\beta_{ii}} \ = \ \sum_{s=i}^{n} m_s \tag{15}$$

Finally, we give the correspondence between the signatures χ and the highest weight Λ. The explicit connection is:

$$n_i = m_i \ , \quad c \ = \ -\tfrac{1}{2}(m_{\tilde\alpha} + m_n) \ = \ -\tfrac{1}{2}(m_1 + \cdots + m_{n-1} + 2m_n) \tag{16}$$

where $\tilde\alpha \ = \ \beta_{11}$ is the highest root.

3. Special reduced multiplets and minimal UIRs

There are several types of multiplets: the main type, (which contains maximal number of ERs/GVMs, the finite-dimensional and the discrete series representations), and various reduced types of multiplets. The multiplets of the main type are in 1-to-1 correspondence with the finite-dimensional irreps of $sp(n, \mathbb{R})$, i.e., they will be labelled by the n positive Dynkin labels $m_i \in \mathbb{N}$. As we mentioned, each main multiplet contains 2^n ERs/GVMs. It is difficult to give explicitly the multiplets for general n. Thus, in the paper [7] we gave for $sp(n, \mathbb{R})$, $n = 6$, the the main type of multiplets and the main reduced types (which depend on $n-1$ parameters). In fact, this gives by reduction also the cases for $n < 6$, since the main multiplet for fixed n coincides with one reduced case for $n + 1$.

In the present paper we give for $n = 2, ..., 6$ the special reduced multiplets which depend on $n-1$ positive integers and one positive odd integer.

3.1. The case sp(2,$\mathbb{R}$)

The material of this subsection is contained in [16] and [17] and is given here to set the stage for the higher rank cases.

The main multiplets R^2_m of $sp(2, \mathbb{R})$ contain $4(= 2^2)$ ERs/GVMs whose signatures were given in [16] in the following pair-wise manner:

$$\begin{aligned} \chi_0^\pm \ &= \ \{\, m_1 \, ; \, \tfrac{3}{2} \pm \tfrac{1}{2}(m_1 + 2m_2) \,\} \\ \chi_a^\pm \ &= \ \{\, m_1 + 2m_2 \, ; \, \tfrac{3}{2} \pm \tfrac{1}{2} m_1 \,\} \end{aligned} \tag{17}$$

The multiplets are given explicitly in Fig. 1. where we use the notation: $\Lambda^\pm = \Lambda(\chi^\pm)$. Each intertwining differential operator is represented by an arrow accompanied by a symbol $i_{j...k}$ encoding the root $\beta_{j...k}$ and the number $m_{\beta_{j...k}}$ which is involved in the BGG criterion. This notation is used to save space, but it can be used due to the fact that only intertwining differential operators which are non-composite are displayed, and that the data β, m_β, which is involved in the embedding $V^\Lambda \longleftrightarrow V^{\Lambda - m_\beta,\beta}$ turns out to involve only the m_i corresponding to simple roots, i.e., for each β, m_β there exists $i = i(\beta, m_\beta, \Lambda) \in \{1, \ldots, 2n-1\}$, such that $m_\beta = m_i$. Hence the data $\beta_{j...k}$, $m_{\beta_{j...k}}$ is represented by $i_{j...k}$ on the arrows.

The pairs $\Lambda^\pm$ of (26) in Figure 1. are symmetric w.r.t. to the bullet in the middle of the figure - this represents the Weyl symmetry realized by two Knapp-Stein integral operators (9):

$$G^\pm_{KS} \ : \ \mathcal{C}_{\Lambda^\mp} \ \longrightarrow \ \mathcal{C}_{\Lambda^\pm} \ . \tag{18}$$

In [16] and [17] the same multiplet was given with the Knapp-Stein operators displayed explicitly - see Figure 2. The differential operators are denoted by arrows, the integral operators - by dashed arrows. We see that the Knapp-Stein operator from $\mathcal{C}_{\Lambda^-_a}$ to $\mathcal{C}_{\Lambda^+_a}$ is degenerated to the differential operator denoted by 1_{12}. Certainly, the latter degeneration is seen already in Figure 1 from the symmetry w.r.t. the bullet in the centre of the figure.

For further use we denote as $\mathcal{D}^\pm$ the invariant subspaces of $\mathcal{C}_{\Lambda^\pm}$ such that $\mathcal{D}^\pm$ is the image of $G^\pm_{KS}$ and the kernel of $G^\mp_{KS}$. This feature is common for all $sp(n)$.

The special reduced multiplets R^2_s also contain 4 ERs/GVMs whose signatures are given in the following pair-wise manner:

$$\begin{aligned} \chi_0^\pm \ &= \ \{\, m_1\,;\ \tfrac{3}{2} \pm \tfrac{1}{2}(m_1+\mu)\,\} \\ \chi_a^\pm \ &= \ \{\, m_1+\mu\,;\ \tfrac{3}{2} \pm \tfrac{1}{2}m_1\,\} \end{aligned} \tag{19}$$

where $\mu \in 2I\!N - 1$. The multiplets are given explicitly in Fig. 3.

Furthermore, as in the main multiplet these representations have all the same Casimirs, but none of them contains a finite-dimensional irrep. Neither the are related as in the quartet since there are no analogs of the operators from χ_0^- to χ_a^- or from χ_a^+ to χ_0^+. In terms of the quartet diagram on Figure 2, only the horizontal lines/arrows remain valid.

Thus, although superficially there are three connected components in the Figure, taking into account the Knapp-Stein operators each multiplet contains two connected components, or submultiplets:

- a doublet consisting of $\Lambda_0^\pm$;
- a doublet consisting of $\Lambda_a^\pm$.

Next we recall from [18] following the exposition in [17] that each ER χ_0^+ of (17) contains both a holomorphic discrete series irrep and its conjugate anti-holomorphic discrete series irrep. The direct sum of the representation spaces of these two irreps is the invariant subspace $\mathcal{D}_0^+$ of $\mathcal{C}_{\Lambda_0^+}$ mentioned above. The statement about the disposition of the (anti)holomorphic discrete series in the the ERs χ_0^+ of the main multiplets is valid for all n. In particular, the corresponding lowest value of the conformal weight is $d = n+1$ which equals the dimension of the corresponding space-time.

For the lack of space we do not discuss the disposition of the (anti)holomorphic discrete series, the first reduction point (FRP), and other positive energy irreps except the minimal UIRs [19]. Here the latter are two special irreps discovered by Dirac [20] and called 'singletons' or 'Di' and 'Rac' in [21]:

$$\text{Rac} \ : \ (d, s_0) \ = \ (1/2, 0) \ , \qquad \text{Di} \ : \ (d, s_0) \ = \ (1, 1/2) \ . \tag{20}$$

Both are situated in the special reduced multiplets: the 'Rac' is situated in the ER/GVM χ_0^- of (19) with $m_1 = \mu = 1$, while the 'Di' in the ER/GVM χ_a^- of (19) with $m_1 = \mu = 1$.

3.2. The case sp(3,$I\!R$)

The main multiplets R_m^3 of $sp(3,I\!R)$ contain $8(=2^3)$ ERs/GVMs whose signatures are given in the following pair-wise manner:

$$\begin{aligned}
\chi_0^\pm \ &= \ \{\, (m_1, m_2)^\pm \,;\, 2 \pm \tfrac{1}{2}(m_{12} + 2m_3) \,\} \\
\chi_a^\pm \ &= \ \{\, (m_1, m_2 + 2m_3)^\pm \,;\, 2 \pm \tfrac{1}{2} m_{12} \,\} \\
\chi_b^\pm \ &= \ \{\, (m_{12}, m_2 + 2m_3)^\pm \,;\, 2 \pm \tfrac{1}{2} m_1 \,\} \\
\chi_c^\pm \ &= \ \{\, (m_2, m_{12} + 2m_3)^\pm \,;\, 2 \mp \tfrac{1}{2} m_1 \,\}
\end{aligned} \tag{21}$$

and the notation $(...)^\pm$ employs the conjugation (2) :

$$(n_1, ..., n_{n-1})^- \ = \ (n_1, ..., n_{n-1}) \ , \qquad (n_1, ..., n_{n-1})^+ \ = \ (n_1, ..., n_{n-1})^*$$

The multiplets are given explicitly in Fig. 4.

The special reduced multiplets R_s^3 also contain 8 ERs/GVMs whose signatures are given in the following pair-wise manner:

$$\begin{aligned}
\chi_0^\pm \ &= \ \{\, (m_1, m_2)^\pm \,;\, 2 \pm \tfrac{1}{2}(m_{12} + \mu) \,\} \\
\chi_a^\pm \ &= \ \{\, (m_1, m_2 + \mu)^\pm \,;\, 2 \pm \tfrac{1}{2} m_{12} \,\} \\
\chi_b^\pm \ &= \ \{\, (m_{12}, m_2 + \mu)^\pm \,;\, 2 \pm \tfrac{1}{2} m_1 \,\} \\
\chi_c^\pm \ &= \ \{\, (m_2, m_{12} + \mu)^\pm \,;\, 2 \mp \tfrac{1}{2} m_1 \,\}
\end{aligned} \tag{22}$$

where $\mu \in 2I\!N - 1$. The multiplets are given explicitly in Fig. 5.

Taking into account the Knapp-Stein operators then each such multiplet contains two connected components, or submultiplets:

- a doublet consisting of $\Lambda_0^\pm$;
- a submultiplet of 6 ERs/GVMs consisting of $\Lambda_a^\pm$, $\Lambda_b^\pm$, $\Lambda_c^\pm$.

There are three minimal UIRs situated in Λ_0^-, Λ_a^- and Λ_c^+ with $m_1 = m_2 = \mu = 1$:
- The one in Λ_0^- has trivial $su(3)$ irrep and $d = \frac{1}{2}$.
- The one in Λ_a^- has three-dimensional $su(3)$ irrep and $d = 1$.
- The one in Λ_c^+ has six-dimensional $su(3)$ irrep and $d = \frac{3}{2}$.

Note that Λ_a^- and Λ_c^+ are in the same connected component - this feature is commented in Section 4.

3.3. The case sp(4,$I\!R$)

The main multiplets R_m^4 of $sp(4,I\!R)$ contain $16(= 2^4)$ ERs/GVMs whose signatures are given in the following pair-wise manner:

$$\begin{aligned}
\chi_0^\pm &= \{ (m_1, m_2, m_3)^\pm \,;\, \pm\tfrac{1}{2}(m_{13} + 2m_4) \} \\
\chi_a^\pm &= \{ (m_1, m_2, m_3 + 2m_4)^\pm \,;\, \pm\tfrac{1}{2}m_{13} \} \\
\chi_b^\pm &= \{ (m_1, m_{23}, m_3 + 2m_4)^\pm \,;\, \pm\tfrac{1}{2}m_{12} \} \\
\chi_c^\pm &= \{ (m_{12}, m_3, m_{23} + 2m_4)^\pm \,;\, \pm\tfrac{1}{2}m_1 \} \\
\chi_d^\pm &= \{ (m_2, m_3, m_{13} + 2m_4)^\pm \,;\, \mp\tfrac{1}{2}m_1 \} \\
\chi_e^\pm &= \{ (m_2, m_3 + 2m_4, m_{13})^\pm \,;\, \mp\tfrac{1}{2}m_1 \} \\
\chi_f^\pm &= \{ (m_{12}, m_3 + 2m_4, m_{23})^\pm \,;\, \pm\tfrac{1}{2}m_1 \} \\
\chi_g^\pm &= \{ (m_1, m_{23} + 2m_4, m_3)^\pm \,;\, \pm\tfrac{1}{2}m_{12} \}
\end{aligned} \tag{23}$$

The multiplets are given explicitly in Fig. 3.

The special reduced multiplets R_s^4 also contain 16 ERs/GVMs whose signatures are given in the following pair-wise manner:

$$\begin{aligned}
\chi_0^\pm &= \{ (m_1, m_2, m_3)^\pm \,;\, \tfrac{5}{2} \pm \tfrac{1}{2}(m_{13} + \mu) \} \\
\chi_a^\pm &= \{ (m_1, m_2, m_3 + \mu)^\pm \,;\, \tfrac{5}{2} \pm \tfrac{1}{2}m_{13} \} \\
\chi_b^\pm &= \{ (m_1, m_{23}, m_3 + \mu)^\pm \,;\, \tfrac{5}{2} \pm \tfrac{1}{2}m_{12} \} \\
\chi_c^\pm &= \{ (m_{12}, m_3, m_{23} + \mu)^\pm \,;\, \tfrac{5}{2} \pm \tfrac{1}{2}m_1 \} \\
\chi_d^\pm &= \{ (m_2, m_3, m_{13} + \mu)^\pm \,;\, \tfrac{5}{2} \mp \tfrac{1}{2}m_1 \} \\
\chi_e^\pm &= \{ (m_2, m_3 + \mu, m_{13})^\pm \,;\, \tfrac{5}{2} \mp \tfrac{1}{2}m_1 \} \\
\chi_f^\pm &= \{ (m_{12}, m_3 + \mu, m_{23})^\pm \,;\, \tfrac{5}{2} \pm \tfrac{1}{2}m_1 \} \\
\chi_g^\pm &= \{ (m_1, m_{23} + \mu, m_3)^\pm \,;\, \tfrac{5}{2} \pm \tfrac{1}{2}m_{12} \}
\end{aligned} \tag{24}$$

where $\mu \in 2I\!N - 1$. The multiplets are given explicitly in Fig. 3s.

Taking into account the Knapp-Stein operators then each such multiplet contains three connected components, or submultiplets:
- a doublet consisting of $\Lambda_0^\pm$;

- a submultiplet of 8 ERs/GVMs consisting of $\Lambda_a^\pm$, $\Lambda_b^\pm$, $\Lambda_c^\pm$, $\Lambda_d^\pm$;
- a submultiplet of 6 ERs/GVMs consisting of $\Lambda_e^\pm$, $\Lambda_f^\pm$, $\Lambda_g^\pm$.

There are four minimal UIRs situated in Λ_0^-, Λ_a^-, Λ_d^+, and Λ_g^- with $m_1 = m_2 = m_3 = \mu = 1$:

- The one in Λ_0^- has have trivial $su(4)$ irrep and $d = \frac{1}{2}$.
- The one in Λ_a^- has fundamental $su(4)$ irrep and $d = 1$.
- The one in Λ_g^- has 20-dimensional $su(4)$ irrep and $d = \frac{3}{2}$.
- The one in Λ_d^+ has another 20-dimensional $su(4)$ irrep and $d = 2$.

Note that Λ_a^- and Λ_d^+ are in the same connected component.

3.4. The case sp(5,$I\!R$)

The main multiplets R_m^5 of $sp(5,I\!R)$ contain $32 (= 2^5)$ ERs/GVMs whose signatures were given in [7]. The special reduced multiplets R_s^5 also contain 32 ERs/GVMs whose signatures are given in the following pair-wise manner:

$$\begin{aligned}
\chi_0^\pm &= \{\, (m_1, m_2, m_3, m_4)^\pm \,;\, \pm\tfrac{1}{2}(m_{14} + \mu) \,\} \\
\chi_a^\pm &= \{\, (m_1, m_2, m_3, m_4 + \mu)^\pm \,;\, \pm\tfrac{1}{2} m_{14} \,\} \\
\chi_b^\pm &= \{\, (m_1, m_2, m_{34}, m_4 + \mu)^\pm \,;\, \pm\tfrac{1}{2} m_{13} \,\} \\
\chi_c^\pm &= \{\, (m_1, m_{23}, m_4, m_{34} + \mu)^\pm \,;\, \pm\tfrac{1}{2} m_{12} \,\} \\
\chi_{c'}^\pm &= \{\, (m_1, m_2, m_{34} + \mu, m_4)^\pm \,;\, \pm\tfrac{1}{2} m_{13} \,\} \\
\chi_d^\pm &= \{\, (m_{12}, m_3, m_4, m_{24} + \mu)^\pm \,;\, \pm\tfrac{1}{2} m_1 \,\} \\
\chi_{d'}^\pm &= \{\, (m_1, m_{23}, m_4 + \mu, m_{34})^\pm \,;\, \pm\tfrac{1}{2} m_{12} \,\} \\
\chi_e^\pm &= \{\, (m_2, m_3, m_4, m_{14} + \mu)^\pm \,;\, \mp\tfrac{1}{2} m_1 \,\} \\
\chi_{e'}^\pm &= \{\, (m_{12}, m_3, m_4 + \mu, m_{24})^\pm \,;\, \pm\tfrac{1}{2} m_1 \,\} \\
\chi_{e''}^\pm &= \{\, (m_1, m_{24}, m_4 + \mu, m_3)^\pm \,;\, \pm\tfrac{1}{2} m_{12} \,\} \\
\chi_f^\pm &= \{\, (m_2, m_3, m_4 + \mu, m_{14})^\pm \,;\, \mp\tfrac{1}{2} m_1 \,\} \\
\chi_{f'}^\pm &= \{\, (m_{12}, m_{34}, m_4 + \mu, m_{23})^\pm \,;\, \pm\tfrac{1}{2} m_1 \,\} \\
\chi_{f''}^\pm &= \{\, (m_1, m_{24} + \mu, m_4, m_3)^\pm \,;\, \pm\tfrac{1}{2} m_{12} \,\} \\
\chi_g^\pm &= \{\, (m_2, m_{34}, m_4 + \mu, m_{13})^\pm \,;\, \mp\tfrac{1}{2} m_1 \,\} \\
\chi_{g'}^\pm &= \{\, (m_{12}, m_{34} + \mu, m_4, m_{23})^\pm \,;\, \pm\tfrac{1}{2} m_1 \,\} \\
\chi_h^\pm &= \{\, (m_2, m_{34} + \mu, m_4, m_{13})^\pm \,;\, \mp\tfrac{1}{2} m_1 \,\}
\end{aligned} \tag{25}$$

where $\mu \in 2I\!N - 1$. The multiplets are given explicitly in Fig. 2s.

Taking into account the Knapp-Stein operators then each such multiplet contains three connected components, or submultiplets:

- a doublet consisting of $\Lambda_0^\pm$;
- a submultiplet of 10 ERs/GVMs starting with Λ_a^-;
- a submultiplet of 20 ERs/GVMs starting with $\Lambda_{c'}^-$.

There are five minimal UIRs situated in Λ_0^-, Λ_a^-, Λ_e^+, $\Lambda_{c'}^-$ and $\Lambda_{f''}^-$ with $m_1 = m_2 = m_3 = m_4 = \mu = 1$:

- The one in Λ_0^- has have trivial $su(5)$ irrep and $d = \frac{1}{2}$.
- The one in Λ_a^- has fundamental $su(5)$ irrep and $d = 1$.
- The one in $\Lambda_{c'}^-$ has 50-dimensional $su(5)$ irrep and $d = \frac{3}{2}$.
- The one in $\Lambda_{f''}^-$ has 175-dimensional $su(5)$ irrep and $d = 2$.
- The one in Λ_e^+ has 70-dimensional $su(5)$ irrep and $d = \frac{5}{2}$.

Note that Λ_a^- and Λ_e^+ are in the same connected component, same for $\Lambda_{c'}^-$ and $\Lambda_{f''}^-$.

3.5. The case sp(6,$I\!R$)

The main multiplets R_m^6 of $sp(6, I\!R)$ contain $64 (= 2^6)$ ERs/GVMs whose signatures were given in [7] and we omit here for the lack of space. The special reduced multiplets R_s^6 also contain 64 ERs/GVMs whose signatures are given in the following pair-wise manner:

$$\begin{aligned}
\chi_0^\pm &= \{ (m_1, m_2, m_3, m_4, m_5)^\pm ; \pm\tfrac{1}{2}(m_{15} + \mu) \} \qquad (26)\\
\chi_a^\pm &= \{ (m_1, m_2, m_3, m_4, m_5 + \mu)^\pm ; \pm\tfrac{1}{2}m_{15} \}\\
\chi_b^\pm &= \{ (m_1, m_2, m_3, m_{45}, m_5 + \mu)^\pm ; \pm\tfrac{1}{2}m_{14} \}\\
\chi_c^\pm &= \{ (m_1, m_2, m_{34}, m_5, m_{45} + \mu)^\pm ; \pm\tfrac{1}{2}m_{13} \}\\
\chi_{c'}^\pm &= \{ (m_1, m_2, m_3, m_{45} + \mu, m_5)^\pm ; \pm\tfrac{1}{2}m_{14} \}\\
\chi_d^\pm &= \{ (m_1, m_{23}, m_4, m_5, m_{35} + \mu)^\pm ; \pm\tfrac{1}{2}m_{12} \}\\
\chi_{d'}^\pm &= \{ (m_1, m_2, m_{34}, m_5 + \mu, m_{45})^\pm ; \pm\tfrac{1}{2}m_{13} \}\\
\chi_e^\pm &= \{ (m_{12}, m_3, m_4, m_5, m_{25} + \mu)^\pm ; \pm\tfrac{1}{2}m_1 \}\\
\chi_{e'}^\pm &= \{ (m_1, m_{23}, m_4, m_5 + \mu, m_{35})^\pm ; \pm\tfrac{1}{2}m_{12} \}\\
\chi_{e''}^\pm &= \{ (m_1, m_2, m_{35}, m_5 + \mu, m_4)^\pm ; \pm\tfrac{1}{2}m_{13} \}\\
\chi_f^\pm &= \{ (m_2, m_3, m_4, m_5, m_{15} + \mu)^\pm ; \mp\tfrac{1}{2}m_1 \}\\
\chi_{f'}^\pm &= \{ (m_{12}, m_3, m_4, m_5 + \mu, m_{25})^\pm ; \pm\tfrac{1}{2}m_1 \}\\
\chi_{f''}^\pm &= \{ (m_1, m_{23}, m_{45}, m_5 + \mu, m_{34})^\pm ; \pm\tfrac{1}{2}m_{12} \}\\
\chi_{f'''}^\pm &= \{ (m_1, m_2, m_{35} + \mu, m_5, m_4)^\pm ; \pm\tfrac{1}{2}m_{13} \}\\
\chi_g^\pm &= \{ (m_2, m_3, m_4, m_5 + \mu, m_{15})^\pm ; \mp\tfrac{1}{2}m_1 \}\\
\chi_{g'}^\pm &= \{ (m_{12}, m_3, m_{45}, m_5 + \mu, m_{24})^\pm ; \pm\tfrac{1}{2}m_1 \}\\
\chi_{g''}^\pm &= \{ (m_1, m_{23}, m_{45} + \mu, m_5, m_{34})^\pm ; \pm\tfrac{1}{2}m_{12} \}\\
\chi_h^\pm &= \{ (m_2, m_3, m_{45}, m_5 + \mu, m_{14})^\pm ; \mp\tfrac{1}{2}m_1 \}\\
\chi_{h'}^\pm &= \{ (m_{12}, m_3, m_{45} + \mu, m_5, m_{24})^\pm ; \pm\tfrac{1}{2}m_1 \}\\
\chi_{h''}^\pm &= \{ (m_2, m_3, m_{45} + \mu, m_5, m_{14})^\pm ; \mp\tfrac{1}{2}m_1 \}\\
\chi_j^\pm &= \{ (m_2, m_{34}, m_5, m_{45} + \mu, m_{13})^\pm ; \mp\tfrac{1}{2}m_1 \}\\
\chi_{j'}^\pm &= \{ (m_{12}, m_{34}, m_5, m_{45} + \mu, m_{23})^\pm ; \pm\tfrac{1}{2}m_1 \}
\end{aligned}$$

$$\begin{aligned}
\chi^{\pm}_{j''} &= \{\,(m_1, m_{24}, m_5, m_{45}+\mu, m_3)^{\pm}\,;\ \pm\tfrac{1}{2}m_{12}\,\}\\
\chi^{\pm}_{k} &= \{\,(m_2, m_{34}, m_5+\mu, m_{45}, m_{13})^{\pm}\,;\ \mp\tfrac{1}{2}m_{1}\,\}\\
\chi^{\pm}_{k'} &= \{\,(m_{12}, m_{34}, m_5+\mu, m_{45}, m_{23})^{\pm}\,;\ \pm\tfrac{1}{2}m_{1}\,\}\\
\chi^{\pm}_{k''} &= \{\,(m_1, m_{24}, m_5+\mu, m_{45}, m_3)^{\pm}\,;\ \pm\tfrac{1}{2}m_{12}\,\}\\
\chi^{\pm}_{\ell} &= \{\,(m_2, m_{35}, m_5+\mu, m_4, m_{13})^{\pm}\,;\ \mp\tfrac{1}{2}m_{1}\,\}\\
\chi^{\pm}_{\ell'} &= \{\,(m_{12}, m_{35}, m_5+\mu, m_4, m_{23})^{\pm}\,;\ \pm\tfrac{1}{2}m_{1}\,\}\\
\chi^{\pm}_{\ell''} &= \{\,(m_1, m_{25}, m_5+\mu, m_4, m_3)^{\pm}\,;\ \pm\tfrac{1}{2}m_{12}\,\}\\
\chi^{\pm}_{m} &= \{\,(m_2, m_{35}+\mu, m_5, m_4, m_{13})^{\pm}\,;\ \mp\tfrac{1}{2}m_{1}\,\}\\
\chi^{\pm}_{m'} &= \{\,(m_{12}, m_{35}+\mu, m_5, m_4, m_{23})^{\pm}\,;\ \pm\tfrac{1}{2}m_{1}\,\}\\
\chi^{\pm}_{m''} &= \{\,(m_1, m_{25}+\mu, m_5, m_4, m_3)^{\pm}\,;\ \pm\tfrac{1}{2}m_{12}\,\}
\end{aligned}$$

where $\mu \in 2I\!N - 1$.

Taking into account the Knapp-Stein operators then each such multiplet contains four connected components, or submultiplets:

- a doublet consisting of $\Lambda^{\pm}_0$;
- a submultiplet of 12 ERs/GVMs starting with Λ^-_a ;
- a submultiplet of 30 ERs/GVMs starting with $\Lambda^-_{c'}$;
- a submultiplet of 20 ERs/GVMs starting with $\Lambda^-_{f'''}$.

There are six minimal UIRs with $m_1 = m_2 = m_3 = m_4 = m_5 = \mu = 1$:

- The one in Λ^-_0 has have trivial $su(6)$ irrep and $d = \frac{1}{2}$.
- The one in Λ^-_a has fundamental $su(6)$ irrep and $d = 1$.
- The one in $\Lambda^-_{c'}$ has 105-dimensional $su(6)$ irrep and $d = \frac{3}{2}$.
- The one in $\Lambda^-_{f'''}$ has 980-dimensional $su(6)$ irrep and $d = 2$.
- The one in $\Lambda^-_{m''}$ has 1764-dimensional $su(6)$ irrep and $d = \frac{5}{2}$.
- The one in Λ^+_f has 252-dimensional $su(6)$ irrep and $d = 3$.

Note that Λ^-_a and Λ^+_f are in the same connected component, same for $\Lambda^-_{c'}$ and $\Lambda^-_{m''}$.

4. Summary, discussion of results and outlook

Each special reduced multiplet of $sp(n, I\!R)$ contains $[\frac{n}{2}] + 1$ connected components, or submultiplets, although if we do not take into account the Knapp-Stein operators then there would be $n + 1$ connected components.

All the ERs/GVMs in a special reduced multiplet have the same Casimirs for fixed parameters $m_1, ..., m_{n-1}, \mu$, since such a multiplet can be obtained from a main multiplet (of only one connected component) depending on the parameters $m_1, ..., m_{n-1}, m_n$, by replacing the even number $2m_n$ with the odd number μ.

There are n minimal UIRs for $sp(n, I\!R)$ with conformal weights $d = \frac{1}{2}, 1, ..., \frac{n}{2}$ whose corresponding ERs are denoted in the corresponding figures as $\chi^-_0, \chi^-_a, \ldots$. Note that in each case the Knapp-Stein operator acting from the ER containing a minimal UIR degenerates to a differential operator of degree $n, n-1, \ldots, 1$, (respectively to the above enumeration). Note further, that there is no differential operator with image that is an ER containing a minimal UIR - that can be used as an equivalent definition of a minimal UIR. The only operator that acts to such an ER is the conjugate non-degenerate Knapp-Stein integral operator.

Note that only the two singletons with $d = \frac{1}{2}, 1$ are isolated points below the continuous unitary spectrum.

We should also note that the multiplets and intertwining differential operators for $Sp(2r, I\!R)$ are valid for $Sp(r, r)$, though the representation content is different [22].

In the present paper we continued the programme outlined in [2] on the example of the non-compact group $Sp(n, I\!R)$ (started already in [7]). Similar explicit descriptions are planned for other non-compact groups from which we have considered so far the cases of $E_{7(-25)}$ [3],[1] $E_{6(-14)}$ [24], $SU(n, n)$ [25], $SO(p, q)$ [26] as parabolically related to $SO(n, 2)$ [22]. We plan also to extend these considerations to the supersymmetric cases and also to the quantum group setting following as here the procedure of [11] and its generalizations, cf., e.g., [27]. Such considerations are expected to be very useful for applications to string theory and integrable models, cf., e.g., [28].

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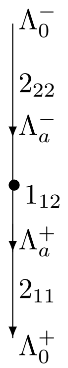

Fig. 1. Main multiplets R_m^2 for $Sp(2,\mathbb{R})$

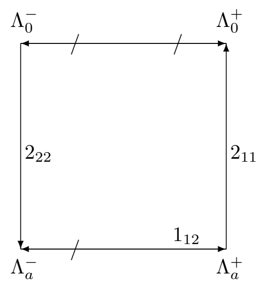

Fig. 2. Main multiplets R_m^2 for $Sp(2,\mathbb{R})$ in quartet view
(arrows are differential operators, dashed arrows are integral operators)

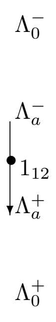

Fig. 3. Special reduced multiplets R_s^2 for $Sp(2,\mathbb{R})$

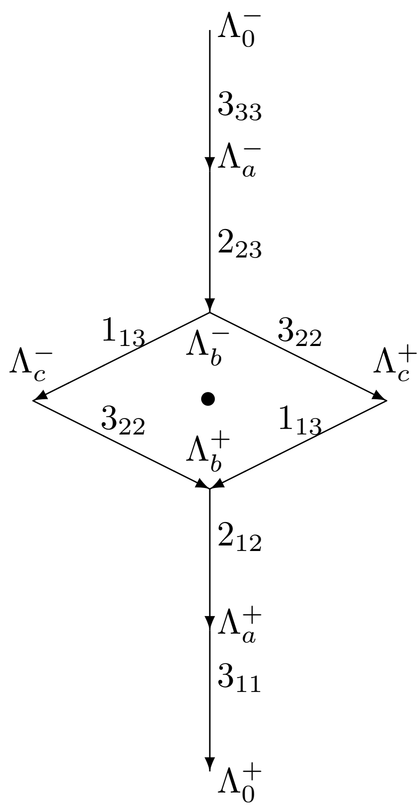

Fig. 4. Main multiplets for $Sp(3, I\!R)$

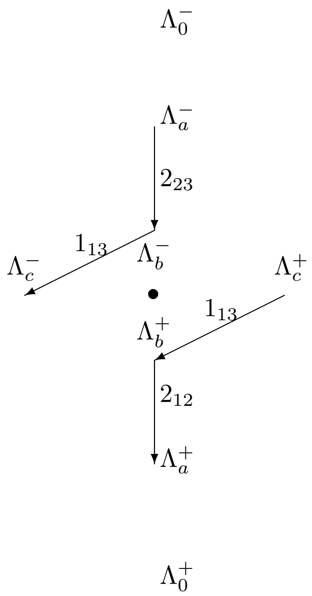

Fig. 5. Special reduced multiplets R_s^3 for $Sp(3,\mathbb{R})$

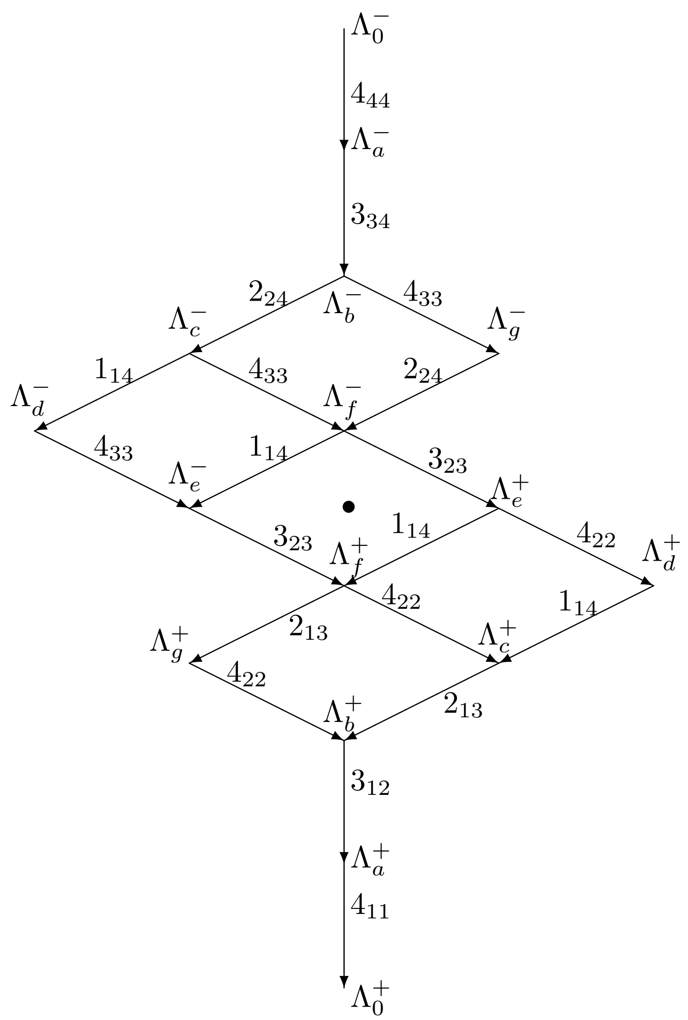

Fig. 6. Main multiplets R_m^4 for $Sp(4, I\!R)$

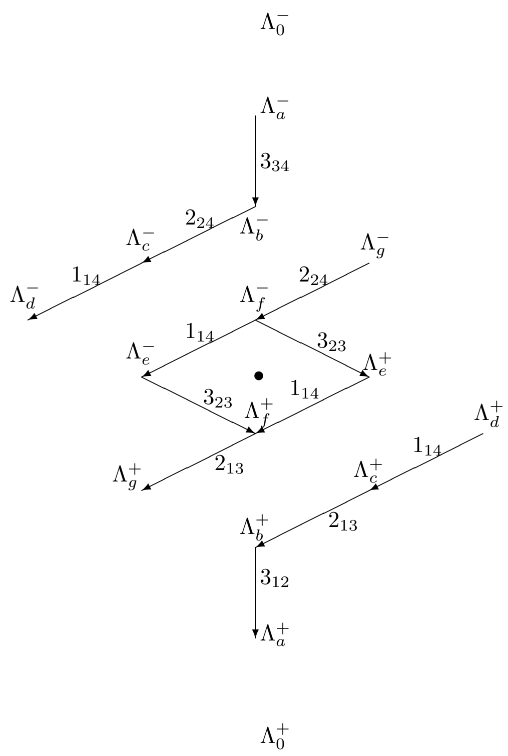

Fig. 7. Special reduced multiplets R_s^4 for $Sp(4, I\!R)$

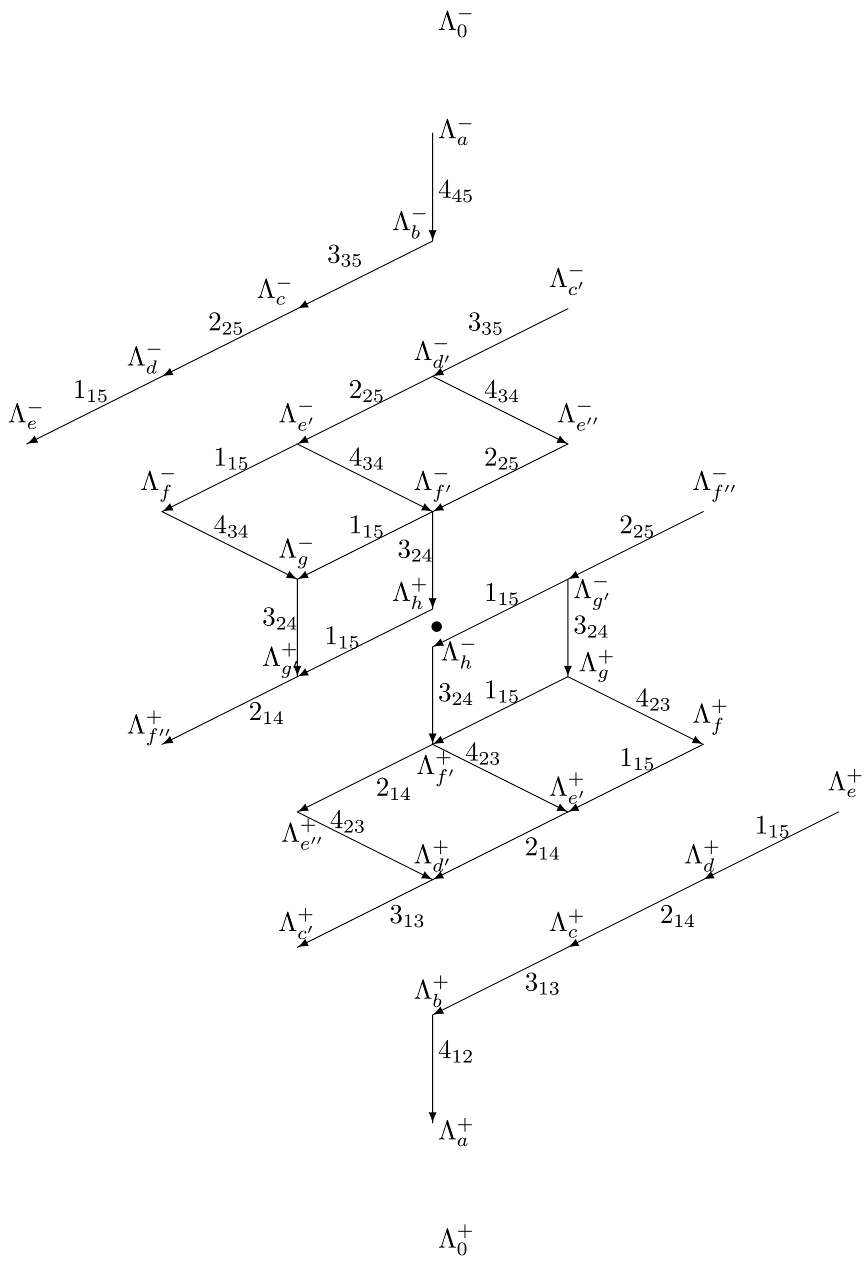

Fig. 8. Special reduced multiplets R_s^5 for $Sp(5, I\!R)$

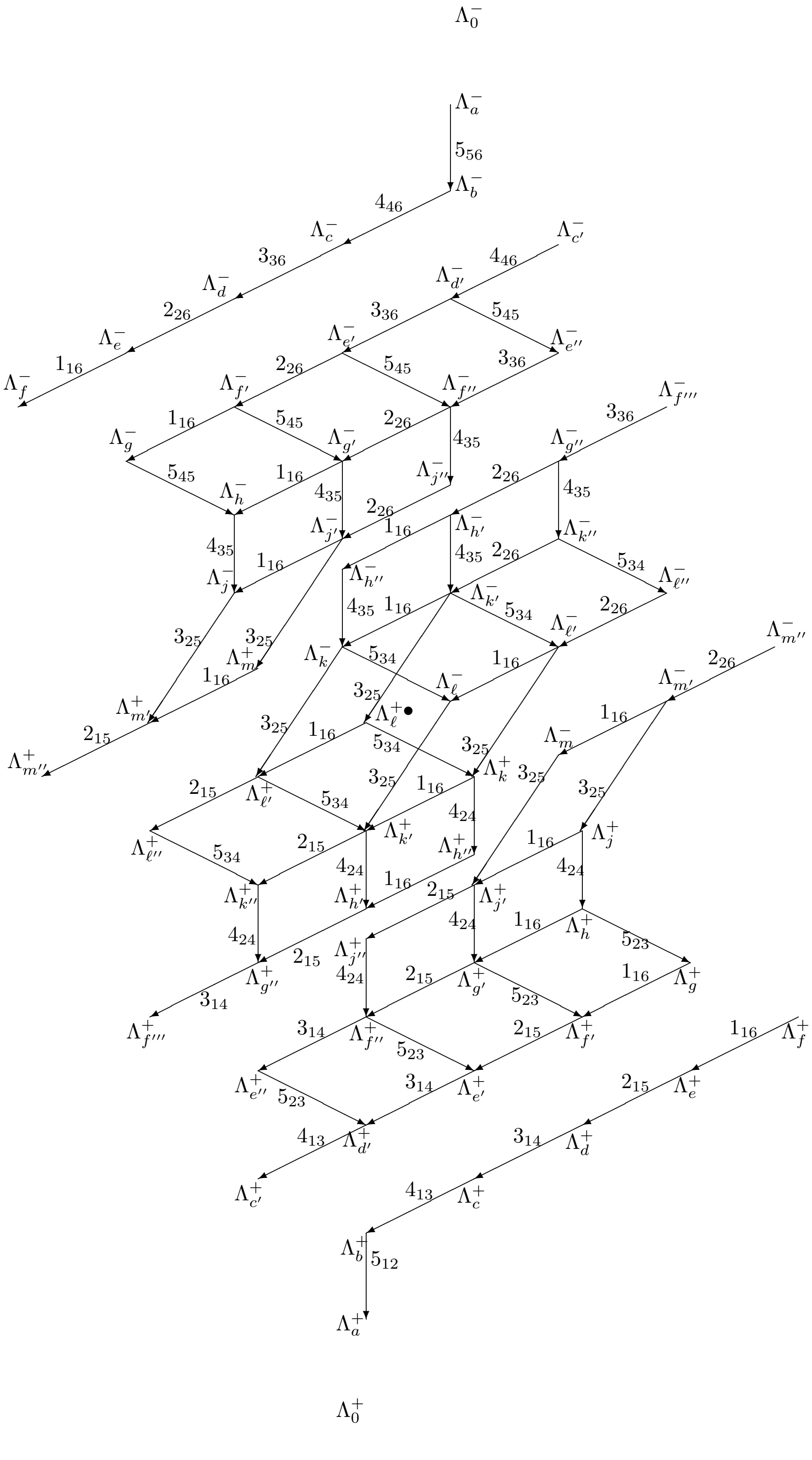

Fig. 9. Special reduced multiplets R_s^6 for $Sp(6,I\!R)$