T-duality in coordinate dependent background *

Ljubica Davidović[†]
Institute of Physics, University of Belgrade, SERBIA

Branislav Sazdović[‡]
Institute of Physics, University of Belgrade, SERBIA

Abstract

We give a prescription for performing the T-dualization for the theories with the coordinate dependent backgrounds. We consider the simplest case of the coordinate dependent background, *the weakly curved background*, which consists of constant metric and Kalb-Ramond field with infinitesimally small coordinate dependent part. T-dual theory is defined in the non-geometric double space, described by the Lagrange multiplier y_μ and its T-dual $\tilde{y}_\mu$. We also demonstrate that the prescription is applicable in the opposite direction as well. This is nontrivial because the T-dual string does not propagate in the weakly curved background.

1. Introduction

T-duality is long investigated property of string theories. It was for the first time described in the context of toroidal compactification in [1]. The majority of papers addressing T-duality considers the string moving in the constant background. In these papers, the prescriptions for the construction of the T-dual theories were established.

In Buscher's construction of T-dual theory [2, 3], one starts with the manifold containing metric $G_{\mu\nu}$, antisymmetric field $B_{\mu\nu}$ and dilaton field Φ. It is required that the metric admits at least one continuous abelian isometry which leaves the action for the σ-model invariant. The covariant Buscher's construction consists of the following steps. First, the isometry is gauged by introducing the gauge fields v^μ_α. Second, the physical equivalence is preserved by introducing the Lagrange multiplier term, which constrains the gauge field strength

$$F^\mu_{\alpha\beta} = \partial_\alpha v^\mu_\beta - \partial_\beta v^\mu_\alpha \tag{1}$$

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† e-mail address: ljubica@ipb.ac.rs

‡ e-mail address: sazdovic@ipb.ac.rs

to zero, making the gauge fields nonphysical. The integration over the Lagrange multipliers y_μ, in the gauged fixed Lagrangian simply recovers the original theory. The integration over the gauge fields v^μ_α, produces the T-dual theory.

In the present article, we consider the weakly curved background and allow the background fields to depend on the coordinates along which we perform duality transformations. The variation with respect to the argument of the background field $B_{\mu\nu}$, produces the topological term, and consequently the isometry is still the symmetry. Our procedure for obtaining the T-dual action is the following:

1. Replace the ordinary derivatives $\partial_\alpha x^\mu$ with the covariant ones $D_\alpha x^\mu = \partial_\alpha x^\mu + v^\mu_\alpha$, where v^μ_α are the gauge fields.
2. Replace the argument of the background fields with the invariant one, substituting every coordinate with its invariant generalization defined by
$$\Delta x^\mu_{inv} = \int d\xi^\alpha D_\alpha x^\mu = x^\mu - x^\mu(\xi_0) + \Delta V^\mu[v_+, v_-], \tag{2}$$
where ΔV^μ is a line integral of the gauge fields v^μ_α.
3. Add Lagrange multiplier term $y_\mu F^\mu_{01}$ and fix the gauge taking $x^\mu(\xi) = x^\mu(\xi_0)$.
4. On the equations of motion for the Lagrange multiplier y_μ the original theory will be obtained.
5. The T-dual theory ${}^\star S[y]$ is obtained by integrating with respect to gauge fields v^μ_α.

Note that the line integral and consequently the invariant coordinate Δx^μ_{inv} and ΔV^μ are path dependent. The Lagrange multiplier term $y_\mu F^\mu_{01}$ grantees that the gauge field is closed $(dv = 0)$ but one should consider the topological contribution as well.

We apply our procedure once again, starting from the T-dual action. The T-dual theory is defined in doubled space but is still globally invariant under the shift of the T-dual coordinate y_μ. Gauging this symmetry, we show that T-dual of the T-dual is indeed the original theory.

2. Bosonic string in the weakly curved background

Let us consider the action [4]
$$S[x] = \kappa \int_\Sigma d^2\xi \ \partial_+ x^\mu \Pi_{+\mu\nu}[x] \partial_- x^\nu, \tag{3}$$
describing the propagation of the bosonic string in the non-trivial background, defined by the space-time metric $G_{\mu\nu}$ and the Kalb-Ramond field $B_{\mu\nu}$
$$\Pi_{\pm\mu\nu}[x] = B_{\mu\nu}[x] \pm \frac{1}{2} G_{\mu\nu}[x]. \tag{4}$$

The integration goes over two-dimensional world-sheet Σ parameterized by $\xi^{\pm} = \frac{1}{2}(\tau \pm \sigma)$. The action is given in the conformal gauge $g_{\alpha\beta} = e^{2F}\eta_{\alpha\beta}$, where $g_{\alpha\beta}$ is the intrinsic world-sheet metric. Here $x^\mu(\xi),\ \mu = 0, 1, ..., D-1$ are the coordinates of the D-dimensional space-time, $\kappa = \frac{1}{2\pi\alpha'}$, and $\partial_\pm = \partial_\tau \pm \partial_\sigma$.

The consistency of the theory requires that the background fields satisfy space-time equations of motion

$$R_{\mu\nu} - \frac{1}{4} B_{\mu\rho\sigma} B_{\nu}{}^{\rho\sigma} = 0\,, \quad D_\rho B^{\rho}{}_{\mu\nu} = 0, \tag{5}$$

where $B_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}$ is the field strength of the field $B_{\mu\nu}$, and $R_{\mu\nu}$ and D_μ are Ricci tensor and covariant derivative with respect to space-time metric. We consider the weakly curved background, defined by the following expressions

$$G_{\mu\nu} = const, \quad B_{\mu\nu}[x] = b_{\mu\nu} + \frac{1}{3} B_{\mu\nu\rho} x^\rho = b_{\mu\nu} + h_{\mu\nu}. \tag{6}$$

which satisfies the space-time equations of motion (5), if the constant $B_{\mu\nu\rho}$ is taken to be infinitesimally small and all the calculations are done in the first order in $B_{\mu\nu\rho}$.

3. Generalized Bouscher's construction

The standard Bouscher's construction of T-dual theory, is applied to the target space with isometries. Despite of x^μ-dependence of the background fields, the weakly curved background preserves the global shift symmetry

$$\delta x^\mu = \lambda^\mu = const, \tag{7}$$

for the closed string. For simplicity we assume that all the coordinates are compact.

As $B_{\mu\nu}$ is linear in coordinate, the variation of the action is proportional to the total divergence

$$\delta S = \frac{\kappa}{3} B_{\mu\nu\rho} \lambda^\rho \epsilon^{\alpha\beta} \int d^2\xi \partial_\alpha (x^\mu \partial_\beta x^\nu) = 0, \tag{8}$$

which vanishes in the case of the closed string and the topologically trivial mapping of the world-sheet into the space-time.

3.1. Gauging shift symmetry

In comparison to the standard Boucher construction, the procedure is changed, because of the coordinate dependence of the fields. As usual, to localize the global shift symmetry, we introduce the gauge fields v^μ_α which transform as

$$\delta v^\mu_\alpha = -\partial_\alpha \lambda^\mu, \quad (\lambda^\mu = \lambda^\mu(\tau, \sigma)), \tag{9}$$

and substitute the ordinary derivatives with the covariant ones

$$\partial_\alpha x^\mu \to D_\alpha x^\mu = \partial_\alpha x^\mu + v^\mu_\alpha. \tag{10}$$

In the weakly curved background, this replacement is, however, not sufficient to make the action locally invariant. Because of this, we additionally replace the coordinate x^μ, with the invariant coordinate defined by

$$\Delta x^\mu_{inv} \equiv \int_P d\xi^\alpha \, D_\alpha x^\mu = x^\mu - x^\mu(\xi_0) + \Delta V^\mu, \tag{11}$$

where

$$\Delta V^\mu \equiv \int_P d\xi^\alpha v^\mu_\alpha. \tag{12}$$

The path P is taken from $\xi^\alpha_0(\tau_0, \sigma_0)$ to $\xi^\alpha(\tau, \sigma)$. The path dependence will be discussed in 3.2. for the world-sheets with trivial holonomies and in 6. for world-sheets with the nontrivial ones.

The main requirement is that the dual theory is equivalent to the initial one. So, in order to make the degrees of freedom originating from the gauge fields nonphysical, the corresponding field strength

$$F^\mu_{\alpha\beta} \equiv \partial_\alpha v^\mu_\beta - \partial_\beta v^\mu_\alpha, \tag{13}$$

must vanish. We can achieve this by introducing the Lagrange multiplier y_μ, and the appropriate term in the Lagrangian

$$S_{inv} = \kappa \int d^2\xi \Big[D_+ x^\mu \Pi_{+\mu\nu}[\Delta x_{inv}] D_- x^\nu + \frac{1}{2}(v^\mu_+ \partial_- y_\mu - v^\mu_- \partial_+ y_\mu) \Big], \tag{14}$$

where the last term is equal $\frac{1}{2} y_\mu F^\mu_{+-}$ up to the total divergence. Fixing the gauge $x^\mu(\xi) = x^\mu(\xi_0)$ we obtain

$$S_{fix}[y, v_\pm] = \kappa \int d^2\xi \Big[v^\mu_+ \Pi_{+\mu\nu}[\Delta V] v^\nu_- + \frac{1}{2}(v^\mu_+ \partial_- y_\mu - v^\mu_- \partial_+ y_\mu) \Big], \tag{15}$$

where y_μ and $v^\mu_\pm$ are independent variables and ΔV^μ is defined in (12).

3.2. Integrating out the Lagrange multiplier

Let us show that the gauge fixed action (15) is equivalent to the initial one (3). The equation of motion with respect to the Lagrange multiplier y_μ, enforces the field strength of the gauge fields to vanish

$$\partial_+ v^\mu_- - \partial_- v^\mu_+ = 0. \tag{16}$$

Its solution

$$v^\mu_\pm = \partial_\pm x^\mu, \tag{17}$$

substituted into (12) gives

$$\Delta V^\mu(\xi) = x^\mu(\xi) - x^\mu(\xi_0). \tag{18}$$

Let us stress that the value of the ΔV^μ does not depend on the choice of the path P. Using Stoke's theorem the defining integral along the closed path P, can be rewritten as the integral over the surface S which spans the path $P = \partial S$,

$$\oint_{P=\partial S} d\xi^\alpha v_\alpha^\mu = \int_S d^2\xi \ (\partial_+ v_-^\mu - \partial_- v_+^\mu). \tag{19}$$

The equation of motion with respect to y_μ forces this field strength to vanish.

Omitting $x^\mu(\xi_0)$, because the action does not depend on the constant shift of the coordinate we find

$$S_{fix}[v_\pm = \partial_\pm x] \quad = \quad \kappa \int d^2\xi \ \partial_+ x^\mu \Pi_{+\mu\nu}[x] \partial_- x^\nu, \tag{20}$$

which is just the initial action (3).

4. T-dual action in the weakly curved background

The T-dual action can be obtained by eliminating the auxiliary gauge fields from (15). Because V^μ is function of independent variables v_+^μ and v_-^μ, the variation by $v_\pm^\mu$ gives two equations of motion

$$\Pi_{\mp\mu\nu}[\Delta V] v_\pm^\nu + \frac{1}{2}\partial_\pm y_\mu = \mp\beta_\mu^\mp[V], \tag{21}$$

which can be rewritten as

$$v_\pm^\mu(y) \quad = \quad -\kappa\, \Theta_\pm^{\mu\nu}[\Delta V(y)] \Big[\partial_\pm y_\nu \pm 2\beta_\nu^\mp[V(y)]\Big], \tag{22}$$

where

$$\Theta_\pm^{\mu\nu}[\Delta V] = -\frac{2}{\kappa}(G_E^{-1}\Pi_\pm G^{-1})^{\mu\nu} = \theta^{\mu\nu}[\Delta V] \mp \frac{1}{\kappa}(G_E^{-1})^{\mu\nu}[\Delta V], \tag{23}$$

and $G_{\mu\nu}^E \equiv [G - 4BG^{-1}B]_{\mu\nu}$, $\theta^{\mu\nu} \equiv -\frac{2}{\kappa}(G_E^{-1}BG^{-1})^{\mu\nu}$ are the open string background fields: the effective metric and the non-commutativity parameter respectively. The terms

$$\beta_\mu^\alpha[V] \equiv \partial_\mu B_{\nu\rho}\epsilon^{\alpha\beta} V^\nu \partial_\beta V^\rho, \tag{24}$$

come from the variation with respect to $\Delta V^\mu(\xi)$, but depend just on V^μ. After one partial integration, we have

$$\delta_V S_{fix} = -\kappa \int d^2\xi \, \beta^\alpha_\mu[V] \, \partial_\alpha \delta V^\mu = -\kappa \int d^2\xi \beta^\alpha_\mu[V] \delta v^\mu_\alpha. \quad (25)$$

Substituting (22) into the action (15), we obtain T-dual action

$$^\star S[y] \equiv S_{fix}[y] = \frac{\kappa^2}{2} \int d^2\xi \; \partial_+ y_\mu \Theta^{\mu\nu}_-[\Delta V(y)] \partial_- y_\nu, \quad (26)$$

where we neglected the second order term $\beta^-_\mu \beta^+_\nu$.

Note that (22) is not the solution of (21), because V^μ and $\beta^\pm_\mu$ depend on $v^\mu_\pm$. In the general case, the solution for $v^\mu_\pm$ and ΔV^μ can not be trivially found. In the next subsections they will be found in the order needed for the case of the weakly curved background. Finally, to obtain the explicit T-dual action we should substitute the solution for ΔV^μ expressed in terms of y_μ into (26) .

4.1. The case of the flat background (zeroth order iteration)

In the case of the constant background $B_{\mu\nu\rho} = 0$, one has

$$G_{\mu\nu}[x] \to G_{\mu\nu}, \quad B_{\mu\nu}[x] \to b_{\mu\nu}, \quad (27)$$

and all the background fields will be denoted by index 0. As $\Pi_{0+\mu\nu}$ is constant, $\beta^\pm_\mu$ vanishes and (22) has the solution

$$v^{(0)\mu}_\pm = -\kappa \, \Theta^{\mu\nu}_{0\pm} \partial_\pm y_\nu, \quad (28)$$

and the T_0-dual action is

$$S[y] = \frac{\kappa^2}{2} \int d^2\xi \; \partial_+ y_\mu \Theta^{\mu\nu}_{0-} \partial_- y_\nu. \quad (29)$$

Using (12) and (28) we obtain $\Delta V^{(0)\mu} = V^{(0)\mu}(\xi) - V^{(0)\mu}(\xi_0)$ with

$$V^{(0)\mu}(\xi) = -\kappa \theta^{\mu\nu}_0 y_\nu + (g^{-1})^{\mu\nu} \tilde{y}_\nu = (g^{-1})^{\mu\nu}[(2bG^{-1})_\nu{}^\rho y_\rho + \tilde{y}_\nu], \quad (30)$$

where

$$\Delta y_\mu(\xi) \equiv \int_P (d\tau \dot{y}_\mu + d\sigma y'_\mu) = y_\mu(\xi) - y_\mu(\xi_0). \quad (31)$$

and

$$\Delta \tilde{y}_\mu(\xi) \equiv \int_P (d\tau y'_\mu + d\sigma \dot{y}_\mu) = \tilde{y}_\mu(\xi) - \tilde{y}_\mu(\xi_0). \quad (32)$$

4.2. The case of the weakly curved background

Note that the variable V^μ, appears always in the terms containing the infinitesimal $B_{\mu\nu\rho}$. So, as we are working up to the first order in $B_{\mu\nu\rho}$, the zeroth order value $V^{(0)\mu}$, will be substituted in all the expressions and in the rest of the paper the index (0) will be omitted. Finding the expression for V^μ, we in fact solved the eq. (22). The solution is

$$v^\mu_\pm = -\kappa \Theta^{\mu\nu}_\pm[\Delta V]\Big[\partial_\pm y_\nu \pm 2\beta^\mp_\nu[V]\Big], \quad V^\mu(\xi) = -\kappa\theta_0^{\mu\nu} y_\nu + (g^{-1})^{\mu\nu}\tilde{y}_\nu, \tag{33}$$

and the T-dual action (26) takes the form

$$^\star S[y] = \frac{\kappa^2}{2}\int d^2\xi\ \partial_+ y_\mu \Theta^{\mu\nu}_-[\Delta V]\partial_- y_\nu. \tag{34}$$

Comparing the initial action (3) with the T-dual one (34), we see that they are equal under following transformations

$$\partial_\pm x^\mu \to \partial_\pm y_\mu, \qquad \Pi_{+\mu\nu}[x] \to \frac{\kappa}{2}\Theta^{\mu\nu}_-[\Delta V], \tag{35}$$

which implies

$$\begin{aligned} G_{\mu\nu} &\to {}^\star G^{\mu\nu}[y,\tilde{y}] = (G_E^{-1})^{\mu\nu}[\Delta V], \\ B_{\mu\nu}[x] &\to {}^\star B^{\mu\nu}[y,\tilde{y}] = \frac{\kappa}{2}\theta^{\mu\nu}[\Delta V], \\ \Delta V^\mu &= -\kappa\theta_0^{\mu\nu}\Delta y_\nu + (g^{-1})^{\mu\nu}\Delta\tilde{y}_\nu. \end{aligned} \tag{36}$$

Comparing the solutions (33) and (17), we obtain the T-dual transformation of the variables law

$$\partial_\pm x^\mu \cong -\kappa\Theta^{\mu\nu}_\pm[\Delta V]\Big[\partial_\pm y_\nu \pm 2\beta^\mp_\nu[V]\Big]. \tag{37}$$

Let us underline that in the initial theory the metric tensor is constant and the Kalb-Ramond field is linear in coordinate x^μ. In the T-dual theory, both background fields depend on ΔV^μ, which is the linear combination of y_μ and its dual $\tilde{y}_\mu$ and consequently T-dual action is not defined on the geometrical space (defined by the coordinate y_μ) but on the so called doubled target space [8] composed of both y_μ and $\tilde{y}_\mu$.

5. From T-dual to the original theory

The T-dual theory (34) is by construction physically equivalent to the initial one (3). So, we should expect that the T-dual of the T-dual theory is just the initial theory. But, in T-dual theory both T-dual metric tensor $^\star G_{\mu\nu}$ and Kalb-Ramond field $^\star B_{\mu\nu}$ are coordinate dependent. Moreover, they depend on both y_μ and $\tilde{y}_\mu$.

To demonstrate the physical equivalence, we should first find the global symmetry of the T-dual action. Note that the action is not invariant under the constant shift of the argument of $\Theta_-^{\mu\nu}$. But, the transformation

$$\delta y_\mu = \lambda_\mu = const, \tag{38}$$

leaves the argument itself, $\Delta V^\mu = V^\mu(\xi) - V^\mu(\xi_0)$, unchanged and consequently the action (34) is invariant too.

5.1. Gauging the symmetry

Let us localize this symmetry and find the corresponding locally invariant action. The procedure is the same as in subsec. 3.1., the only difference is that here we deal with the double space defined by two coordinates y_μ and $\tilde{y}_\mu$.

We covariantize the derivatives

$$D_\pm y_\mu = \partial_\pm y_\mu + u_{\pm\mu}, \tag{39}$$

introducing the gauge fields $u_{\pm\mu}$ which transform as

$$\delta u_{\pm\mu} = -\partial_\pm \lambda_\mu(\tau,\sigma). \tag{40}$$

The dual background fields argument ΔV^μ is not locally invariant. So, first we construct the invariant expressions for both variables y_μ and $\tilde{y}_\mu$

$$\begin{aligned} \Delta y_\mu^{inv} &\equiv = \int_P (d\tau D_0 y_\mu + d\sigma D_1 y_\nu) = \Delta y_\mu + \Delta U_\mu, \\ \Delta \tilde{y}_\mu^{inv} &\equiv \int_P (d\tau D_1 y_\mu + d\sigma D_0 y_\nu) = \Delta \tilde{y}_\mu + \Delta \tilde{U}_\mu, \end{aligned} \tag{41}$$

where Δy_μ and $\Delta \tilde{y}_\mu$ are defined in (31) and (32) and

$$\Delta U_\mu \equiv \int_P (d\tau u_{0\mu} + d\sigma u_{1\mu}), \quad \Delta \tilde{U}_\mu \equiv \int_P (d\tau u_{1\mu} + d\sigma u_{0\mu}). \tag{42}$$

Now, it is easy to find the generalization of the background fields argument

$$\begin{aligned} \Delta V^\mu_{inv} &\equiv -\kappa\theta_0^{\mu\nu} \Delta y_\nu^{inv} + (g^{-1})^{\mu\nu} \Delta \tilde{y}_\nu^{inv} \\ &= -\kappa\theta_0^{\mu\nu}(\Delta y_\rho + \Delta U_\rho) + (g^{-1})^{\mu\nu}(\Delta \tilde{y}_\nu + \Delta \tilde{U}_\nu) \\ &= \Delta V^\mu[y] + \Delta V^\mu[U], \end{aligned} \tag{43}$$

which is invariant by construction.

Finally, we can construct the dual invariant action

$$^\star S_{inv} = \frac{\kappa}{2} \int d^2\xi \Big[\kappa D_+ y_\mu \Theta_-^{\mu\nu}[\Delta V_{inv}] D_- y_\nu + u_{+\mu}\partial_- z^\mu - u_{-\mu}\partial_+ z^\mu\Big], \tag{44}$$

where the second term makes the gauge fields $u_{\pm\mu}$ nonphysical. The gauge fixing $y_\mu(\xi) = y_\mu(\xi_0)$, produces $D_\pm y_\mu = u_{\pm\mu}$ and $\Delta V^\mu[y] = 0$, so the action becomes

$$^\star S_{fix}[z, u_\pm] = \frac{\kappa}{2} \int d^2\xi \Big[\kappa u_{+\mu} \Theta_-^{\mu\nu}\big[\Delta V[U]\big] u_{-\nu} + u_{+\mu}\partial_- z^\mu - u_{-\mu}\partial_+ z^\mu\Big]. \tag{45}$$

5.2. Integrating out the Lagrange multiplier

The equation of motion with respect to the Lagrange multiplier z^μ

$$\partial_+ u_{-\mu} - \partial_- u_{+\mu} = 0, \tag{46}$$

has the solution

$$u_{\pm\mu} = \partial_\pm y_\mu, \tag{47}$$

which substituted to (42) gives $\Delta U_\mu = \Delta y_\mu$. So, the action (45) on this solution becomes

$$^\star S_{fix}[u_\pm = \partial_\pm y] = \frac{\kappa^2}{2} \int d^2\xi \partial_+ y_\mu \Theta_-^{\mu\nu}\big[\Delta V[y]\big]\partial_- y_\nu, \tag{48}$$

and coincides with the T-dual action (34).

5.3. Integrating out the gauge fields

By varying the action (45), with respect to the gauge fields $u_{\pm\mu}$, using the fact that

$$\Theta_-^{\nu\rho} = \Theta_{0-}^{\nu\rho} - 2\kappa[\Theta_{0-} h \Theta_{0-}]^{\nu\rho}, \tag{49}$$

we obtain the equations of motion

$$\partial_\pm z^\mu = -\kappa\Theta_\pm^{\mu\nu}\big[\Delta V[U]\big]\Big[u_{\pm\nu} \pm 2\beta_\nu^\mp\big[V[U]\big]\Big]. \tag{50}$$

Using the expression $\Theta_\pm^{\mu\nu}\Pi_{\mp\nu\rho} = \frac{1}{2\kappa}\delta_\rho^\mu$, we can extract $u_{\pm\mu}$

$$u_{\pm\mu} = -2\Pi_{\mp\mu\nu}\big[\Delta V[U]\big]\partial_\pm z^\nu \mp 2\beta_\mu^\mp\big[V[U]\big]. \tag{51}$$

Similarly as in the subsection 4.2., we will solve equations (51) and (42) iteratively. From the zeroth order solution of (51) one finds the zeroth order values of U_μ and $\tilde{U}_\mu$

$$U_\mu = -2b_{\mu\nu}z^\nu + G_{\mu\nu}\tilde{z}^\nu, \quad \tilde{U}_\mu = -2b_{\mu\nu}\tilde{z}^\nu + G_{\mu\nu}z^\nu, \tag{52}$$

and confirms that

$$V^\mu[U] = (g^{-1})^{\mu\nu}[2b_\nu{}^\rho U_\rho + \tilde{U}_\nu] = z^\mu, \tag{53}$$

and consequently $\beta^{\pm}_{\mu}[V[U]] = \beta^{\pm}_{\mu}[z]$. Substituting (53) into (51), we obtain its solution

$$u_{\pm\mu} = -2\Pi_{\mp\mu\nu}[\Delta z]\partial_{\pm}z^{\nu} \mp 2\beta^{\mp}_{\mu}[z], \quad \big(\Delta z^{\mu} = z^{\mu}(\xi) - z^{\mu}(\xi_0)\big). \tag{54}$$

Substituting it into the action (45), we obtain

$$^{\star}S_{fix}[z] = \kappa \int d^2\xi \partial_{+}z^{\mu}\Pi_{+\mu\nu}[z(\xi) - z(\xi_0)]\partial_{-}z^{\nu}. \tag{55}$$

But, this action is invariant under the global shift in the coordinate and we can omit the term $z(\xi_0)$ and obtain the T-dual of the T-dual action

$$^{\star\star}S[z] \equiv {}^{\star}S_{fix}[z] = \kappa \int d^2\xi \partial_{+}z^{\mu}\Pi_{+\mu\nu}[z]\partial_{-}z^{\nu}, \tag{56}$$

which is in fact the initial action. So, the second T-duality turns the doubled target space $(y_\mu, \tilde{y}_\mu)$ back to the conventional space z^{μ}.

Comparing (54) with (47), we obtain the T-duality transformation of the variables law

$$\partial_{\pm}y_{\mu} \cong -2\Pi_{\mp\mu\nu}[\Delta z]\partial_{\pm}z^{\nu} \mp 2\beta^{\mp}_{\mu}[z]. \tag{57}$$

Note that this is the inverse transformation of (37). More precisely, substituting y_μ from (57) into (37) one has $\partial_{\pm}x^{\mu} = \partial_{\pm}z^{\mu}$.

6. Global features in the quantum theory

Let us shortly discuss some global features of our procedure. In the classical theory, the invariant coordinate Δx_{inv} is multivalued, and in the quantum theory the holonomies of the world-sheet gauge fields introduce the new obstructions.

For simplicity we will consider the case when the world-sheet is a torus. After the Wick rotation $\tau \to -i\tau$, the term in the action which contains metric tensor $G_{\mu\nu}$ acquires multiplier i, while the terms which contain Kalb-Ramond field $B_{\mu\nu}$ and Lagrange multiplier y_μ remain unchanged. We simplified notation using differential forms and omitting the space-time index μ. The Hodge duality operator is denoted by star. The Euclidean path integral partition function is therefore

$$Z = \int \mathcal{D}y\mathcal{D}v\, e^{-S(v,\Delta V)+i\kappa\int_{\Sigma} vdy}, \tag{58}$$

where

$$S(v, V) = \frac{\kappa}{2}\int_{\Sigma} v\, G^{\star}v - i\kappa\int_{\Sigma} vB[\Delta V]v. \tag{59}$$

We will compare this partition function with one of the original theory.

Let us make the Hodge decomposition of the forms v, dy and dx

$$v = dv_e + d^\dagger v_{ce} + v_h, \quad dy = dy_e + y_h, \quad dx = dx_e + x_h. \tag{60}$$

The 1-form v is separated into exact (v_e is single valued function), co-exact and the harmonic ($dv_h = 0 = d^\dagger v_h$) parts, while the closed 1-forms dy and dx have only the exact and the harmonic parts. The integration with respect to y_e in (58) forces the field strength of the gauge field to vanish due to the appearance of the $\delta(dv)$ which also causes the path independence of ΔV^μ. Using $dv = 0$ and the Riemann bilinear relation, the last term in the exponent becomes

$$\int_\Sigma v y_h = \oint_a v \oint_b y_h - \oint_a y_h \oint_b v, \tag{61}$$

where a and b represent the canonical homology basis for the torus.

All nontrivial holonomies come from the harmonic parts of dy and v, $y_h = y^0_\alpha d\xi^\alpha$, $v_h = v^0_\alpha d\xi^\alpha$. Restricting the coordinate y to periodic one $y \sim y + 2\pi R$, and integrating over y^0_a and y^0_b we obtain

$$Z = \int \mathcal{D}v_e \, dv^0_a dv^0_b \sum_{n_a \in \mathbb{Z}} \delta\Big(\frac{R v^0_b}{\alpha'} - n_a\Big) \sum_{n_b \in \mathbb{Z}} \delta\Big(\frac{R v^0_a}{\alpha'} - n_b\Big)\, e^{-S(v,\Delta V)}. \tag{62}$$

Let us at this point, confirm that ΔV^μ does not depend on the choice of the path P. Let P_1 be some other path with the same initial ξ^α_0 and the final point ξ^α as the path P. Then, the difference in ΔV^μ along closed curve PP_1^{-1}, homological to a curve $m_a a + m_b b$, $(m_a, m_b \in \mathbb{Z})$, is the integral of the harmonic form

$$\Delta V[P](\xi) - \Delta V[P_1](\xi) = \oint_{PP_1^{-1}} v_h = 2\pi(m_a v^0_a + m_b v^0_b). \tag{63}$$

Now, performing the integration over v^0_a and v^0_b in (62), we obtain

$$Z = \int \mathcal{D}v_e \sum_{n_a, n_b \in \mathbb{Z}} e^{-S(v,\Delta V)}, \tag{64}$$

where closed form $v \sim v + 2\pi r$ becomes periodic with

$$r = \frac{\alpha'}{R}. \tag{65}$$

At the same time (63) turns to

$$\Delta V[P] = \Delta V[P_1] + 2\pi r k, \quad (k = m_a n_b + m_b n_a \in \mathbb{Z}). \tag{66}$$

So, variable ΔV^μ is periodic, with the same period r as the x coordinate. Therefore, the only trace of the path dependence of V^μ is its winding.

Substituting $v_e \to x_e$, $v_h \to x_h$ we obtain the initial theory

$$Z \to \int \mathcal{D}x_e \sum_{n_a,n_b\in\mathbb{Z}} e^{-S(dx,x)} = \int \mathcal{D}x \, e^{-S[x]} = Z_0 \tag{67}$$

with $x \sim x + 2\pi r$.

Therefore, the winding modes of the Lagrange multiplier y^μ act as the Lagrange multipliers for the holonomies.

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