String field theory and distribution theory*

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Abstract

In this contribution I review the main results concerning analytic lump solutions in open bosonic string field theory and discuss a mathematical problem linked to their definition. I show that it is possible to eliminate any ambiguity by interpreting the solution string field in a context of extended distribution theory.

1. Introduction

Recently there has been step forward in the search for analytic solutions in cubic open string field theory (OSFT) à la Witten, [5]. Following an earlier suggestion of [14], a general method has been proposed, [1], to find new exact analytic solutions, and in particular solutions that describe inhomogeneous tachyon condensations. On general grounds it is expected that an OSFT defined on a particular boundary conformal field theory (BCFT) has classical solutions describing other boundary conformal field theories [6, 7]. Analytic solutions have actually been constructed describing the tachyon vacuum [8, 15, 9]. But exact solutions describing inhomogeneous and relevant boundary deformations of the initial BCFT were not known until recently, though their existence was predicted [6, 7]. This absence was filled up in [14, 1], and in [2, 3] the energy of a D24-brane solution was calculated for the first time. In [4] these results were extended to analytic SFT solutions corresponding to D(25-p)-branes, for any p, and their energy was calculated.

In section 2 of this contribution I will review the results of [1, 2]. However I will take advantage of these proceedings to deal with a mathematical problem one often meets in the search for analytic SFT solutions, but has not been yet treated in a satisfactory way. A particular version of this question was raised in [3] and a partial solution to it was proposed in appendix D of [2]. Several formal aspects related to it where addressed in [11]. In this note we would like to return to this issue in a more general framework, that of distribution theory. The problem manifests itself, for instance, as a

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would-be violation of the SFT equation of motion for the string field candidates considered in [1, 2, 4, 3], which originates from the use of a Schwinger parametrization of inverse elements. We have argued elsewhere that, when the issue is considered in the proper setting, no violations to the equation of motion occur for the solutions considered in [2, 4]. The offending term, when inserted in convergent integrals, gives vanishing contributions and, on the other hand, can give nonvanishing (but ambiguous) contributions only if inserted in discontinuous integrals (see below for the precise meaning). This suggests that the appropriate mathematical setting to interpret them is the theory of distributions. We suggest that the lump solutions must be considered as distributions. Once this is done, any ambiguity linked to spurious terms in the equation of motion disappears. In this contribution we wish to set the stage for a rigorous treatment of this problem.

The paper is organized as follows. After reviewing the results of [1] and [2] in section 2, in section 3 we outline the problem that arises when we represent $\frac{1}{K+\phi_u}$ by means of a Schwinger parametrization. Section 4 is a pedagogical review of the main ideas in (ordinary) distribution theory which we intend to extend from functions to string fields. In section 5 the case of Fock space is discussed and in section 6 the space of good string test states is introduced and its topologies are discussed. In section 7 we introduce the dual space, i.e. the space of linear functionals, which is proposed as the correct space where the string fields of SFT belong.

2. Review of results on analytic lumps

In [1], to start with, the well-known K, B, c algebra, where K, B, c are defined by

$$K = \frac{\pi}{2} K_1^L |I\rangle, \qquad B = \frac{\pi}{2} B_1^L |I\rangle, \qquad c = c\left(\frac{1}{2}\right)|I\rangle, \tag{1}$$

was enlarged as follows. In the sliver frame (obtained by mapping the UHP to an infinite cylinder C_2 of circumference 2, by the sliver map $\tilde{z} = \frac{2}{\pi}\arctan z$), by adding a (relevant) matter operator

$$\phi = \phi\left(\frac{1}{2}\right)|I\rangle \tag{2}$$

with the properties

$$[c, \phi] = 0, \qquad [B, \phi] = 0, \qquad [K, \phi] = \partial\phi, \tag{3}$$

In this new algebra Q has the following action:

$$Q\phi = c\partial\phi + \partial c\delta\phi. \tag{4}$$

It can be easily proven that

$$\psi_\phi = c\phi - \frac{1}{K+\phi}(\phi - \delta\phi)Bc\partial c \tag{5}$$

does indeed satisfy the OSFT equation of motion

$$Q\psi_\phi + \psi_\phi\psi_\phi = 0. \tag{6}$$

It is clear that (5) is a deformation of the Erler–Schnabl solution, see [10], which can be recovered for $\phi = 1$.

In order to prove that (5) is a solution, one demands that $(c\phi)^2 = 0$, which requires the OPE of ϕ at nearby points to be not too singular.

Using the K, B, c, ϕ algebra one can show that

$$\mathcal{Q}_{\psi_\phi} \frac{B}{K+\phi} = Q\frac{B}{K+\phi} + \left\{\psi_\phi, \frac{B}{K+\phi}\right\} = 1.$$

So, unless the homotopy–field $\frac{B}{K+\phi}$ is singular, the solution has trivial cohomology, which is the defining property of the tachyon vacuum [14, 15]. On the other hand, in order for the solution to be well defined, the quantity $\frac{1}{K+\phi}(\phi - \delta\phi)$ should be well defined too. Finally, in order to be able to show that (5) satisfies the equation of motion, one needs $K + \phi$ to be invertible.

In full generality we thus have a new nontrivial solution if

1. $\frac{1}{K+\phi}$ is singular, but
2. $\frac{1}{K+\phi}(\phi - \delta\phi)$ is regular and
3. $\frac{1}{K+\phi}(K + \phi) = 1$.

In [1] sufficient conditions for ϕ to comply with the first two requirements were determined. Let us parametrize the worldsheet RG flow, referred to above, by a parameter u, where $u = 0$ represents the UV and $u = \infty$ the IR, and rewrite ϕ as ϕ_u, with $\phi_{u=0} = 0$. Then we require for ϕ_u the following properties under the coordinate rescaling $f_t(z) = \frac{z}{t}$

$$f_t \circ \phi_u(z) = \frac{1}{t}\,\phi_{tu}\left(\frac{z}{t}\right) \tag{7}$$

and, most important, that the partition function

$$g(u) \equiv Tr[e^{-(K+\phi_u)}] = \left\langle e^{-\int_0^1 ds\,\phi_u(s)} \right\rangle_{C_1}, \tag{8}$$

satisfies the asymptotic finiteness condition

$$\lim_{u\to\infty} \left\langle e^{-\int_0^1 ds\,\phi_u(s)} \right\rangle_{C_1} = \texttt{finite}. \tag{9}$$

It was pointed out in [1] that this satisfies the first two conditions above i.e. guarantees not only the regularity of the solution but also its 'non-triviality', in the sense that if this condition is satisfied, it cannot fall in the same class as the ES tachyon vacuum solution. It would seem that the last condition above cannot be satisfied in view of the first. But this is not the case. This is the main issue discussed in sec.3 and 4.

We will consider in the sequel a specific relevant operator ϕ_u and the corresponding SFT solution. This operator generates an exact RG flow studied by Witten in [12], see also [13], and is based on the operator (defined in the cylinder C_T of width T in the arctan frame)

$$\phi_u(s) = u(X^2(s) + 2\ln u + 2A), \tag{10}$$

where A is a constant first introduced in [14]. In C_1 we have

$$\phi_u(s) = u(X^2(s) + 2\ln Tu + 2A) \tag{11}$$

and on the unit disk D,

$$\phi_u(\theta) = u(X^2(\theta) + 2\ln \frac{Tu}{2\pi} + 2A). \tag{12}$$

If we set

$$g_A(u) = \langle e^{-\int_0^1 ds\, \phi_u(s)} \rangle_{C_1} \tag{13}$$

we have

$$g_A(u) = \langle e^{-\frac{1}{2\pi}\int_0^{2\pi} d\theta\, u\left(X^2(\theta)+2\ln\frac{u}{2\pi}+2A\right)} \rangle_D.$$

According to [12],

$$g_A(u) = Z(2u)e^{-2u(\ln\frac{u}{2\pi}+A)}, \tag{14}$$

where

$$Z(u) = \frac{1}{\sqrt{2\pi}}\sqrt{u}\Gamma(u)e^{\gamma u} \tag{15}$$

Requiring finiteness for $u \to \infty$ we get $A = \gamma - 1 + \ln 4\pi$, which implies

$$g_A(u) \equiv g(u) = \frac{1}{\sqrt{2\pi}}\sqrt{2u}\Gamma(2u)e^{2u(1-\ln(2u))} \tag{16}$$

and

$$\lim_{u\to\infty} g(u) = 1. \tag{17}$$

Moreover, as it turns out, $\delta\phi_u = -2u$, and so:

$$\phi_u - \delta\phi_u = u\partial_u\phi_u(s). \tag{18}$$

Therefore the ϕ_u just introduced satisfies all the required properties and consequently $\psi_u \equiv \psi_{\phi_u}$ must represent a D24 brane solution*.

In I the expression for the energy of the lump solution was determined by evaluating a three–point function on the cylinder C_T of circumference T in the arctan frame. It is given by

$$\begin{aligned} E[\psi_u] &= -\frac{1}{6}\langle \psi_u \psi_u \psi_u \rangle \\ &= \frac{1}{6}\int_0^\infty d(2uT)\ (2uT)^2 \int_0^1 dy \int_0^y dx\, \frac{4}{\pi} \sin\pi x \sin\pi y \sin\pi(x-y) \qquad (19) \\ &\cdot g(uT)\Bigg\{ -\Big(\frac{\partial_{2uT} g(uT)}{g(uT)}\Big)^3 + G_{2uT}(2\pi x) G_{2uT}(2\pi(x-y)) G_{2uT}(2\pi y) \\ &-\frac{1}{2}\Big(\frac{\partial_{2uT} g(uT)}{g(uT)}\Big)\Big(G_{2uT}^2(2\pi x) + G_{2uT}^2(2\pi(x-y)) + G_{2uT}^2(2\pi y)\Big)\Bigg\}. \end{aligned}$$

where $G_u(\theta)$ represents the correlator on the boundary, first determined by Witten, [12]:

$$G_u(\theta) = \frac{1}{u} + 2\sum_{k=1}^{\infty} \frac{\cos(k\theta)}{k+u} \qquad (20)$$

Moreover $\mathcal{E}_0(t_1, t_2, t_3)$ represents the ghost three–point function in C_T.

$$\begin{aligned} \mathcal{E}_0(t_1,t_2,t_3) &= \langle Bc\partial c(t_1+t_2)\partial c(t_1)\partial c(0)\rangle_{C_T} \\ &= -\tfrac{4}{\pi} \sin\tfrac{\pi t_1}{T} \sin\tfrac{\pi(t_1+t_2)}{T} \sin\tfrac{\pi t_2}{T}\,. \end{aligned} \qquad (21)$$

Finally, to get (19) a change of variables $(t_1, t_2, t_3) \to (T, x, y)$, where

$$x = \frac{t_2}{T}, \qquad y = 1 - \frac{t_1}{T}.$$

is needed.

The expression (19) has been evaluated in [2]. As it turns out, this expression has a UV ($s \approx 0$, setting $s = 2uT$) singularity, which must be subtracted away. Therefore the result one obtains in general will depend on this subtraction. In [2] it has been pointed out that *a physical significance can be assigned only to a subtraction-independent quantity*, and it has been shown how to define and evaluate such a quantity. First a new solution to the EOM, depending on a regulator ϵ, has been introduced†

$$\psi_u^\epsilon = c(\phi_u + \epsilon) - \frac{1}{K + \phi_u + \epsilon}(\phi_u + \epsilon - \delta\phi_u)Bc\partial c. \qquad (22)$$

*This solution was presented in the VI Mathematical Physics Meeting, Belgrade 2010.

†In [2] ψ_u^ϵ was called ψ_ϵ.

its energy being 0 (after the same UV subtraction as in the previous case) in the $\epsilon \to 0$ limit. Then, using it, a solution to the EOM at the tachyon condensation vacuum has been obtained. The equation of motion at the tachyon vacuum is

$$\mathcal{Q}\Phi + \Phi\Phi = 0, \quad \text{where} \quad \mathcal{Q}\Phi = Q\Phi + \psi_u^\epsilon \Phi + \Phi \psi_u^\epsilon. \tag{23}$$

One can easily show that

$$\Phi_0^\epsilon = \psi_u - \psi_u^\epsilon \tag{24}$$

is a solution to (23). The action at the tachyon vacuum is $-\frac{1}{2}\langle \mathcal{Q}\Phi, \Phi\rangle - \frac{1}{3}\langle \Phi, \Phi\Phi\rangle$. Thus the energy of of the lump, $E[\Phi_0]$, is

$$\begin{aligned} E[\Phi_0] &= -\lim_{\epsilon\to 0} \frac{1}{6}\langle \Phi_0^\epsilon, \Phi_0^\epsilon \Phi_0^\epsilon\rangle \qquad (25)\\ &= -\frac{1}{6}\lim_{\varepsilon\to 0}\left[\langle \psi_u, \psi_u\psi_u\rangle - \langle \psi_u^\epsilon, \psi_u^\epsilon\psi_u^\epsilon\rangle - 3\langle \psi_u^\epsilon, \psi_u\psi_u\rangle + 3\langle \psi_u, \psi_u^\epsilon\psi_u^\epsilon\rangle\right]. \end{aligned}$$

The integrals in the four correlators at the RHS, are IR ($s \to \infty$) convergent. The UV subtractions necessary for each correlator are always the same, therefore they cancel out. In [2], after the UV subtraction, the result was

$$\begin{aligned} &-\frac{1}{6}\langle \psi_u, \psi_u\psi_u\rangle = \alpha + \beta, && \lim_{\epsilon\to 0}\langle \psi_u^\epsilon, \psi_u^\epsilon\psi_u^\epsilon\rangle = 0\\ &\frac{1}{6}\lim_{\epsilon\to 0}\langle \psi_u^\epsilon, \psi_u\psi_u\rangle = \alpha - \frac{2}{3}\beta, && \frac{1}{6}\lim_{\epsilon\to 0}\langle \psi_u, \psi_u^\epsilon\psi_u^\epsilon\rangle = \alpha - \frac{1}{3}\beta \end{aligned} \tag{26}$$

where $\alpha + \beta \approx 0.068925$ was evaluated numerically and $\alpha = \frac{1}{2\pi^2}$ was calculated analytically. So $E[\Phi_0] = \alpha$ turns out to be precisely the D24-brane energy. In [4] the same result was extended to any Dp-brane lump.

3. The problem with the Schwinger representation

We now come to the remark of [3] concerning the above solution. In order to obtain (19) one has to use the following Schwinger representation

$$\frac{1}{K + \phi_u} = \int_0^\infty dt\, e^{-t(K+\phi_u)} \tag{27}$$

of the inverse of $K + \phi_u$. When using such a Schwinger representation, however, the identity

$$\frac{1}{K + \phi_u}(K + \phi_u) = I, \tag{28}$$

would seem not to be satisfied. To illustrate the problem, let us calculate the overlap of both the left and the right hand sides of (28) with $Y = \frac{1}{2}\partial^2 c\partial cc$. The right hand side is trivial and, in our normalization, it is

$$\mathrm{Tr}(Y \cdot I) = \lim_{t\to 0}\langle Y(t)\rangle_{C_t}\langle 1\rangle_{C_t} = \frac{V}{2\pi}. \tag{29}$$

To calculate the left hand side we need the Schwinger representation

$$\mathrm{Tr}\big[Y \cdot \frac{1}{K+\phi_u}(K+\phi_u)\big] = \int_0^\infty dt \mathrm{Tr}\big[Y \cdot e^{-t(K+\phi_u)}(K+\phi_u)\big] \tag{30}$$

Making the replacement

$$e^{-t(K+\phi_u)}(K+\phi_u) \to -\frac{d}{dt}e^{-t(K+\phi_u)} \tag{31}$$

one obtains

$$\mathrm{Tr}\big[Y \cdot \frac{1}{K+\phi_u}(K+\phi_u)\big] = g(0) - g(\infty) = \frac{V}{2\pi} - g(\infty), \tag{32}$$

which is different form (29) because $g(\infty)$ is nonvanishing. The latter relation is often written in a stronger form

$$\int_0^\infty dt\, e^{-t(K+\phi_u)}(K+\phi_u) = 1 - \Omega_u^\infty, \qquad \Omega_u^\infty = \lim_{\Lambda\to\infty} e^{-\Lambda(K+\phi_u)} \tag{33}$$

This (strong) equality, however, has to be handled with great care. If the latter is taken literally, we could also write

$$\frac{1}{K+\phi_u} = \int_0^\infty dt\, e^{-t(K+\phi_u)} + \frac{1}{K+\phi_u}\Omega_u^\infty \tag{34}$$

instead of (27). This would imply that eq.(28) is not satisfied, and, consequently, the equation of motion is not satisfied by ψ_u.

It is clear that we need a regularization and the latter can be provided by

$$\lim_{\varepsilon\to 0}\frac{1}{K+\phi+\varepsilon} \tag{35}$$

For $\varepsilon \neq 0$ the inverse of $K+\phi+\varepsilon$ is well defined. The question is now: does the limit (37) exist, and how is it defined? If we take algebraic manipulations at face value, using (35) one gets, [11],

$$Q\psi_u + \psi_u\psi_u = \lim_{\varepsilon\to 0}\left(\frac{\varepsilon}{(K+\phi_u+\varepsilon)}(\phi_u - \delta\phi_u)c\partial c\right)$$

which would imply a violation of the equation of motion.

4. Solving the problem with distributions

The one described above is but an example of a frequently occurring problem in SFT. It concerns the existence and definition of such limits as

$$\lim_{\varepsilon \to 0} \frac{\varepsilon^a}{K+\varepsilon} \tag{36}$$

or

$$\lim_{\varepsilon \to 0} \frac{\varepsilon^a}{K+\phi+\varepsilon} \tag{37}$$

where a is a positive number, $K = K_1^L|I\rangle$ and ϕ is, as above, a 2d matter field. These objects, which are crucial in constructing and discussing analytic solutions of the SFT equation of motion, are not altogether new in the literature. They bring together two old problems. What is new is that these old problems appear simultaneously in a new context, that of SFT. First we remark that both (36) and (37) have 'pointlike support'. Both in fact can be nonzero only in correspondence with one particular value taken by K or $K+\phi$. For instance, in the first case, this corresponds to the zero mode of K_1^L.

The previous expressions are somehow the analog of objects like

$$\lim_{\varepsilon \to 0} \frac{\varepsilon}{x^2+\varepsilon} \tag{38}$$

defined on the real line, which has support at $x = 0$. (38) is well-known to be an ordinary distribution. Distributions may be ordinary functions, but the interesting thing about them is that they allow us to define objects which are almost functions but not quite. In order to be able to evaluate them one has first to define a space of regular (test) functions with its topology and define a rule in order to evaluate the distributions on such test functions. Distributions are in fact linear functionals over the test functions space. Therefore they belong to the dual space. The limit (38) is taken in the dual space, which must therefore be equipped with an adequate topology.

The previous simple considerations are meant to emphasize the fundamental role played by couples of dual spaces in carefully defining ordinary distributions. But, of course, (36) and (37) are no ordinary distributions. The role of points in space for ordinary distributions is played in (36) and (37) by states in first quantized string theory. This is not at all a new problem in quantum mechanics. A similar one was met and solved long ago by means of Gelfand triples (or rigged Hilbert spaces). Let us consider the simple example of a 1d nonrelativistic particle on the real line. The quantum description is obtained by solving the Schroedinger equation with suitable convergence properties at infinity, so that the wave-function ψ is square integrable on the real line. The completion of the space of such functions leads to a Hilbert space $\boldsymbol{\mathcal{H}}$. However the position and momentum operators and polynomials thereof are not well defined on all the functions in the

Hilbert space; for this to be the case one has to single out the subspace $\mathbf{\Phi}$ of smooth functions. This is not enough because neither in $\mathbf{\Phi}$ nor in $\boldsymbol{\mathcal{H}}$ can we accommodate the eigenfunctions of the position and momentum operator. But the latter are appropriately accommodated in the dual space $\mathbf{\Phi}'$ of $\mathbf{\Phi}$, i.e. on the space of linear continuous functionals on $\mathbf{\Phi}$. An important aspects is that the topology of $\mathbf{\Phi}$ has to be chosen in such a way that the application of the position and momentum operators are continuous, which requires a stronger topology than the Hilbert space topology of $\boldsymbol{\mathcal{H}}$. As a consequence the dual space $\mathbf{\Phi}'$ is larger. We have in fact the inclusion

$$\mathbf{\Phi} \subset \boldsymbol{\mathcal{H}} \subset \mathbf{\Phi}'$$

This is called rigged Hilbert space or Gelfand triple. The elements of $\mathbf{\Phi}'$ are often called 'distributions' too.

A similar construction has to be envisaged in order to correctly interpret (36) and (37). The general structure is always the same. We have a space of 'regular' objects, say $\mathcal{R}$ with a suitable topology, its completion $\bar{\mathcal{R}}$ and the dual space $\mathcal{R}'$ of 'distributions', with the inclusion

$$\boldsymbol{\mathcal{R}} \subset \bar{\boldsymbol{\mathcal{R}}} \subset \boldsymbol{\mathcal{R}}' \tag{39}$$

and with the duality rule in order to evaluate elements of $\boldsymbol{\mathcal{R}}'$ on elements of $\boldsymbol{\mathcal{R}}$.

As we shall see, the construction we need is actually a hybrid of both ordinary distributions and rigged Hilbert space.

5. Ordinary distributions

To prepare our main analysis it is worth going through a short introduction of the distribution theory, [22]. The purpose is to extract the essential concepts underlying this theory, to be generalized later on. *Distributions* or *generalized functions* are *almost functions* but not quite. In which sense *almost* must be understood is explained by the following (qualitatively expressed) general result: *Distributions can be reduced to finite order derivatives of locally integrable functions.* An example is the Dirac delta function on the real line, which can be viewed as the derivative of the Heaviside step function. The latter is of course locally integrable.

The reason why we need such objects is that they appear in many physical problems, which cannot be described in terms of ordinary functions.

In order to be able to carefully define distributions one has to view them as linear continuous functionals of a topological vector space formed by ordinary continuous differentiable (up to a certain order) functions with good convergence properties at infinity, hereafter called *test functions.* Defining a functional means defining a (linear) rule that associates a number to any test function. This rule always consists of a Lebesgue integral.

The (vector) space of test functions must be topological, because a topology is necessary in order to tell what test functions are close and what are not.

In this way we can define continuity for functionals. Also the dual space, the space of functionals, must have a topology, because we want to be able to take limits of distributions.

5.1. Example of test function spaces

Test functions will be denoted by Greek letters $\varphi, \psi, ...$ A space of test functions is, for instance, $\mathbf{K}(a)$: it is formed by all the functions on the real line which are infinitely differentiable and have support inside the interval $|x| \leq a$. A linear combination of them has still the same characteristics, therefore $\mathbf{K}(a)$ is a vector space. $\mathbf{K}(a)$ is also a topological vector space, but its topology is quite nontrivial. In fact it is a *countably normed vector space*, that is it is characterized by an infinite sequence of norms. They are defined as follows

$$||\varphi||_p = \max_{|x|\leq a}\{\varphi(x), \varphi'(x), \ldots, \varphi^{(p)}(x)\}, \quad p = 0, 1, 2, \ldots$$

We have of course $||\varphi||_p \leq ||\varphi||_{p+1}$. Since $\mathbf{K}(a)$ is a vector space its topology is defined by a set of neighborhoods of 0. The latter are given by

$$U_{p,\epsilon} = \{||\varphi||_p < \epsilon\}$$

One can prove that this defines a topology.

We can complete $\mathbf{K}(a)$ with respect to one of these norms, say $||\cdot||_p$. In such a way we get $\overline{\mathbf{K}(a)^p} \equiv \mathbf{K}_p(a)$. This is the space of all the functions $\varphi(x)$ with support in $|x| \leq a$ and continuously differentiable up to order p. We have

$$\mathbf{K}_1(a) \supset \mathbf{K}_2(a) \supset \ldots \supset \mathbf{K}(a)$$

Another test function space is $\mathbf{S}$. It is the space of functions $\varphi(x)$ indefinitely differentiable in $\mathbb{R}$, that for $|x| \to \infty$ tend to 0 more rapidly than any power of $1/|x|$. This is also a countably normed space. A third example is the space $\mathbf{Z}(a)$ of entire analytic functions in the complex variable $z = x + iy$, satisfying appropriate inequalities. It is a countably normed space.

All the above test function spaces can be generalized to many variables. They are countably normed. A generic countably normed test function space will be denoted by the symbol $\mathbf{\Phi}$.

5.2. Distributions

Distributions are linear continuous functionals on a test function space $\mathbf{\Phi}$. The space of such functionals (dual space) will be denoted by $\mathbf{\Phi}'$. The rule for evaluating a functional f over a test function φ will be denoted by $f(\varphi) \equiv \langle f, \varphi \rangle$. For instance, for any function $\varphi \in \mathbf{K}(a)$ we can define

$$\langle f, \varphi \rangle = \int_{-a}^{a} \varphi^{(m)}(x) d\mu(x)$$

where m is a fixed positive integer and μ a function with bounded variation. One can prove that f is a distribution. Since $\mathbf{\Phi}$ is countably normed we can define

$$||f||_p = \sup_{||\varphi||_p \leq 1} |\langle f, \varphi \rangle|,$$

which is a norm in the dual space $\mathbf{\Phi}'_p$ of $\mathbf{\Phi}_p$. We have

$$\mathbf{\Phi}' = \bigcup_{p=1}^{\infty} \mathbf{\Phi}'_p$$

that is, $\mathbf{\Phi}'$ is also countably normed and it is the union of an increasing sequence of Banach spaces whose norm is weaker and weaker:

$$\mathbf{\Phi}'_1 \subset \mathbf{\Phi}'_2 \subset \ldots \subset \mathbf{\Phi}'$$

The so-defined topology of the dual is called *strong*. There is also a *weak* topology. It is defined as follows. Take a finite set of test functions $\varphi_1, \ldots \varphi_m$. Then a neighborhood of 0 in $\mathbf{\Phi}'$ is defined by the f's that satisfy

$$|\langle f, \varphi_1 \rangle| < \epsilon, |\langle f, \varphi_2 \rangle| < \epsilon, \ldots |\langle f, \varphi_m \rangle| < \epsilon$$

We have the following definition:

A sequence of elements of $\mathbf{\Phi}'$, f_n, converge weakly to $f \in \mathbf{\Phi}'$ if and only if, for any $\varphi \in \mathbf{\Phi}$ we have

$$\lim_{n \to \infty} \langle f_n, \varphi \rangle = \langle f, \varphi \rangle$$

It so happens that the dual $\mathbf{\Phi}'$ of a countably normed space $\mathbf{\Phi}$ is complete with respect to the weak topology.

For all the spaces $\mathbf{K}(a)$, $\mathbf{S}$ and $\mathbf{Z}$ the following representation theorem holds

Representation theorem. Any distribution f belonging to $\mathbf{\Phi}'$ admit the following representation

$$\langle f, \varphi \rangle = \int dx\, f_0(x) P(D) \varphi(x) \tag{40}$$

where $f_0(x)$ is a locally summable function (Lebesgue integrable function) and $P(D)$ is a polynomial of $D = \frac{d}{dx}$.

A simple example of this theorem is provided by the Dirac delta function

$$\langle \delta_a, \varphi \rangle = -\int dx\, \theta(x-a) \varphi'(x)$$

where θ is the step function.

5.2.1. Examples

An example of problem solved by distribution theory is the inversion of a polynomial $P(z)$ in the complex plane. The problem

$$P(z)f = 1 \tag{41}$$

has always a solution in terms of a distribution in $\mathbf{Z}'$. In particular the function z or $z - z_0$ has an inverse represented by a distribution in $\mathbf{Z}'$.

Among these examples let us recall that distributions can often be defined as limits of ordinary functions. For instance

$$\delta(x) = \lim_{\varepsilon \to 0} \frac{1}{\pi} \frac{\sqrt{\varepsilon}}{x^2 + \varepsilon} \tag{42}$$

This follows from the fact that

$$\int_a^b dx \, \frac{\sqrt{\varepsilon}}{x^2 + \varepsilon} = \frac{1}{\pi} \left(\arctan \frac{b}{\sqrt{\varepsilon}} - \arctan \frac{a}{\sqrt{\varepsilon}} \right), \tag{43}$$

Thus whenever the interval $[a, b]$ includes 0, in the $\varepsilon \to 0$ limit we get 1, while if it does not contain 0 we get 0. This justifies (42), see [22]. In the following we will meet an expression very similar to (42), but with an important difference. The object we will have to discuss is the analogue of

$$\lim_{\varepsilon \to 0} \frac{1}{\pi} \frac{\varepsilon}{x^2 + \varepsilon} \tag{44}$$

Due to the additional $\sqrt{\varepsilon}$ factor in the numerator the analogous integral (43) always vanishes in the $\varepsilon \to 0$ limit, even if the $[a, b]$ interval includes 0. Therefore

$$\lim_{\varepsilon \to 0} \frac{1}{\pi} \frac{\varepsilon}{x^2 + \varepsilon} = 0 \tag{45}$$

as a distribution.

5.3. How large are the test function spaces

It would seem that there is an arbitrariness in the problem we have considered so far. Since a distribution f is determined by its values on test functions, how do we know that this procedure is unique? In other words suppose that for any $\varphi \in \mathbf{\Phi}$ we have $\langle f, \varphi \rangle = 0$, is the space $\mathbf{\Phi}$ *rich* enough for us to conclude that $f = 0$?

This problem is well formulated by the following *definition*: a test space $\mathbf{\Phi}$ is *rich enough* if, for any locally integrable function $f(x)$, existence of the integral

$$\int dx \, f(x) \, \varphi(x), \qquad \forall \varphi \in \mathbf{\Phi}$$

and

$$\int dx\, f(x)\, \varphi(x) = 0, \qquad \forall \varphi \in \mathbf{\Phi}$$

imply that $f(x) = 0$ almost everywhere. This means that f is identically zero if we restrict our consideration to elements of $\mathbf{\Phi}$ alone.

The spirit of this definition is to guarantee that set of test functions is a powerful enough filter that only very 'fine' non-regular behaviors can pass through it. This filter cannot detect for instance functions which are nonvanishing only in a set of measure zero, but it does detect any piecewise regular behaviour and, in particular, any regular behaviour.

All the test spaces considered above are rich enough. Once this condition is satisfied, we shall say that a distribution is zero if it vanishes when contracted with all the elements of $\mathbf{\Phi}$.

Warning. Since the duality rule can be formally extended to a space larger than the space of test functions, one can easily envisage a situation in which a zero distribution when evaluated on a non-test function does not vanish. But of course this is illegal.

6. The dual of the Fock space

We would like to present now an analog of the rigged Hilbert space example introduced in sec.3, based on a Fock space rather than on a function space. Let us consider the string oscillators α_n, with the algebra $[\alpha_m, \alpha_n] = m\delta_{n+m,0}$. We construct the corresponding Fock space $\mathcal{F}$ by acting on the vacuum $|0\rangle$ with the creation operators α_n with $n < 0$. $\mathcal{F}$ is the linear span of all the states of the type

$$|\phi_{n_1,n_2,\dots,n_s}\rangle = \alpha_{-n_1}\alpha_{-n_2}\dots\alpha_{-n_s}|0\rangle, \qquad n_i \in \mathbb{Z}_+ \tag{46}$$

As is well known the oscillator algebra defines a scalar product once we assume that $\langle 0|0\rangle = 1$. This also implies the definition of a norm $||\phi|| = \sqrt{\langle\phi|\phi\rangle}$. The completion of $\mathcal{F}$ with respect to this norm gives rise to a Hilbert space $\mathcal{H} \equiv \overline{\mathcal{F}}$ which contains $\mathcal{F}$ as a dense subset.

Like in the quantum mechanical example in sec. 4, however, this Hilbert space does not contain all the interesting states in SFT. For instance, it does not contain wedge states and, generically, surface states, because such states do not have a finite norm. Similar to the rigged Hilbert space we would like to find a space larger than $\mathcal{H}$. To this end we have to give up the scalar product topology in $\mathcal{F}$ and introduce a stronger one. The linear functionals which are continuous with respect to this stronger topology will form the dual space $\mathcal{F}'$. If suitably chosen the space $\mathcal{F}'$ should allow us to embed surface states and the like.

The Fock space $\mathcal{F}$ plays the role of $\mathcal{R}$, the space of regular objects. Therefore it is natural to say that, if a generic state vanishes when contracted with all Fock space states, it is zero. The limitation to Fock space states is essential, when using this criterion, otherwise it is easy to construct a counterexample. Consider a definite Fock space state ϕ_0. It obviously has finite contractions with all the other Fock space states. Therefore if we consider $\phi_\epsilon = \epsilon\phi_0$, we have that $\lim_{\epsilon\to 0}\phi_\epsilon$ vanishes when contracted with all the states of the Fock space. Thus $\lim_{\epsilon\to 0}\phi_\epsilon = 0$ according to the above criterion. If, on the other hand, we contract ϕ_ϵ with states like $\frac{1}{\epsilon}\phi$, where ϕ is any Fock space state, in the limit $\epsilon \to 0$ we find a finite result. This is not surprising since the norm of $\frac{1}{\epsilon}\phi$ becomes infinite in the $\epsilon \to 0$ limit, thus is does not belong to $\boldsymbol{\mathcal{F}}$.

7. The space of test string fields

7.1. Preliminary discussion

The problem we would like to discuss here is the existence of the inverse of $K+\phi_u$. As we have pointed out $K+\phi = (K_1^L + \phi_u(\frac{1}{2})|I\rangle$, where K_1^L is the left translation operator, a symmetric operator in the Fock space, and $|I\rangle$ is the star algebra identity. The spectrum of $\mathcal{K}_u \equiv K_1^L + \phi_u(\frac{1}{2})$, which is also a symmetric operator, lies in the real axis and is likely to include also the origin. If it does and the identity string field contains the zero mode of $\mathcal{K}_u$, then a problem of invertibility arises. It was shown that the obstruction to invert $K+\phi_u$ is measured by the expression

$$\mathcal{A}_\varepsilon = e^{-\varepsilon\partial_\varepsilon}\frac{\varepsilon}{K+\phi_u+\varepsilon} = \lim_{\varepsilon\to 0}\frac{\varepsilon}{K+\phi_u+\varepsilon} \tag{47}$$

This quantity, whatever it is, can be nonvanishing only where $K+\phi_u$ vanishes, i.e. in correspondence with the zero mode of $\mathcal{K}_u$. $\mathcal{A}_\varepsilon$ has support, if any, only on this zero mode. It is a *distribution-like object* and must be treated within the formalism of distribution theory. Of course the latter must be suitably generalized to the framework of SFT in which instead of a position in space (for instance $r=0$ in ordinary field theory) is replaced by a string configuration (for instance the state representing the zero mode above). The correct evaluation of $\mathcal{A}(\varepsilon)$ is of upmost importance, for a naive manipulation of the equation of motion leads to

$$Q\psi_u + \psi_u\psi_u = \lim_{\varepsilon\to 0}\frac{\varepsilon}{K+\phi_u+\varepsilon}(\phi_u+\delta\phi_u)c\partial c \tag{48}$$

i.e. to an apparent violation of the equation of motion.

Our main claim is that $\mathcal{A}_\varepsilon$, when interpreted in the appropriate setting, is a vanishing object, the zero in distribution theory (so, in particular, the RHS of (48) is actually zero). Let us first come to this conclusion via heuristics.

7.2. A heuristic argument

The term $\mathcal{A}_\varepsilon$ is of the type (42) or (44). Let us recall that the latter is actually 0, see (45). Let us write

$$\mathcal{A}_\varepsilon = \lim_{\varepsilon\to 0}\varepsilon\int_0^\infty dt\, e^{-t(K+\phi_u+\varepsilon)} \tag{49}$$

We can think of replacing $K+\phi_u$ with its eigenvalue κ and integrating over it to simulate the path integration. However, since in general the factor $e^{-t(K+\phi_u+\varepsilon)}$ appears in correlators multiplied by other factors for a more realistic simulation we replace $K+\phi_u$ by κ^a, with $a>1$ (it can only be a power of κ since it must vanish for $\kappa\to 0$). Then we have

$$< \mathcal{A}_\varepsilon > \approx \varepsilon \int_0^\infty dt e^{-t\varepsilon} \int_0^m d\kappa\, e^{-t\kappa^a} \sim \begin{cases} \varepsilon^{\frac{1}{a}} & a>1 \\ \varepsilon \log \varepsilon & a \leq 1 \end{cases} \tag{50}$$

where m is a small but fixed number. Thus $\mathcal{A}_\varepsilon = 0$, at least according to this rough approximation.

In order to properly justify this conclusion we have to go first through the full process of defining the appropriate space of test states and its dual. As we have seen in the previous section, in function theory distributions are the objects of the dual of a topological vector space. Our aim is to interpret $\mathcal{A}_\varepsilon$ as a distribution, i.e. a functional in a suitable topological vector space. But of course this vector space is not the most general one, rather it must have properties that assimilate it to a space of functions, and the duality rule (i.e the rule by which we can evaluate a functional over the test states) had better be an integral. This would allow us to use the analogy with ordinary distribution theory as close as possible. Fortunately this is possible in the present case, thanks to the Schwinger representation of the inverse of $K+\phi_u$:

$$\frac{1}{K+\phi_u} = \int_0^\infty dt\, e^{-t(K+\phi_u)} \tag{51}$$

This representation makes concrete the abstract properties of the functional in question and 'localizes' the zero mode of $\mathcal{K}_u$ at $t=\infty$ (for the representation (51) becomes singular when $K+\phi_u$ vanishes). This 'localization property' makes our life much easier because it allows us to formulate the problem of defining test states, dual functionals and their properties in terms of their t dependence via the Schwinger representation (51).

7.3. Good test string fields

Let us now construct a set of string states that have good properties in view of forming the topological vector space of test states we need for our problem.

We will proceed in a heuristic way. First of all the states we are looking for must be such that the resulting contractions with $\Gamma(\varepsilon) = \mathcal{A}_\varepsilon(\phi_u - \delta\phi_u)c\partial c$ be nonsingular (with respect to singularities due to collapsing points). But, especially, they must be characterized by integrable behaviour in the UV and, ignoring the overall $e^{-\varepsilon t}$ factor, in the IR. The IR corresponds to $t\to\infty$, where, as was noticed above, the zero mode of $\mathcal{K}_u$ is 'localized'. Therefore the IR behaviour will be crucial in our discussion. It is self-evident that *all* the states with such properties annihilate $\Gamma(\varepsilon)$, so it would seem that we are preparing test states that already satisfy the condition we wish to prove. But, on the other hand, the only possibility of getting a nonzero result is linked, as usual, to correlators characterized by IR linearly divergent integrals (without the exponential $e^{-\varepsilon t}$). Now, such integrals are characterized by the fact that their $\varepsilon \to 0$ limit is discontinuous, therefore the corresponding states can hardly be considered *good test states*. The true question we have to ask, then, is whether the good test states we construct are 'enough'.

Consider states created by multiple products of the factor $H(\phi_u, \varepsilon) = \frac{1}{K+\phi_u+\varepsilon}(\phi_u - \delta\phi_u)$ and contract them with $\Lambda(\varepsilon) = \lim_{\varepsilon\to 0} \frac{1}{K+\phi_u+\varepsilon}(\phi_u - \delta\phi_u)Bc\partial c$. From what we said above we are looking for contractions which are finite and whose $\varepsilon \to 0$ limit is continuous. More precisely, let us define

$$\mathbf{\Psi}_n(\phi_u, \varepsilon) = H(\phi_u, \varepsilon)^{n-1} Bc\partial c\, H(\phi_u, \varepsilon), \qquad n \geq 2 \tag{52}$$

Contracting with $\Lambda(\varepsilon)$: $\langle \mathbf{\Psi}_n(\phi_u, \varepsilon)Bc\partial cB, \Lambda(\varepsilon)\rangle$, we obtain a correlator whose IR and UV behaviour (before the the $\varepsilon \to 0$ limit is taken) is not hard to guess. The correlators take the form

$$\int_0^\infty ds\, s^n e^{-\tilde{\eta}s} g(s) \tag{53}$$

$$\times \int \prod_{i=1}^n dx_i\, \mathcal{E} \left(\left(-\frac{\partial g(s)}{g(s)} \right)^{n+1} + \ldots + \left(-\frac{\partial g(s)}{g(s)} \right)^{n-k+1} G_s^k + \ldots + G_s^{n+1} \right)$$

where the notation is the same as in section 2 ($s = 2uT$ and $\tilde{\eta} = \frac{\varepsilon}{2u}$), but we have tried to make it as compact as possible. The angular variables x_i have been dropped in $\mathcal{E}$ and G_s. Using the explicit form of G_s, [1], expanding the latter with the binomial formula and integrating over the angular variables, one gets

$$\int \prod_{i=1}^n dx_i\, \mathcal{E}\, G_s^k = \sum_{l=0}^k \frac{1}{s^{k-l}} \sum_{n_1,\ldots,n_l} \frac{P_l(n_1,\ldots,n_l)}{Q_l(n_1,\ldots,n_l)} \prod_{i=1}^l \frac{1}{p_i(n_1,\ldots,n_l) + s} \tag{54}$$

the label l counts the number of cosine factors in each term. Here n_i are positive integral labels which come from the discrete summation in G_s; $p_i(n_1,\ldots,n_l)$ are polynomials linear in n_i. Next, P_l and Q_l are polynomials in n_i which come from the integration in the angular variables. Every integration in x_i increases by 1 the difference in the degree of Q_l and P_l, so that generically $\mathrm{deg}Q_l - \mathrm{deg}P_l = n$. But in some subcases the integration over angular variables gives rise to Kronecker deltas among the indices, which may reduce the degree of Q_l. So actually the relation valid in all cases is $\mathrm{deg}Q_l \geq \mathrm{deg}P_l$. But one has to take into account that the number of angular variables to be summed over decreases accordingly.

We are now in the condition to analyze the UV behaviour of (53). Let us consider, for instance, the first piece

$$\sim \int_0^\infty ds\, e^{-\tilde{\eta}s} s^n\, g(\frac{s}{2}) \left(\frac{\partial_s g(\frac{s}{2})}{g(\frac{s}{2})} \right)^{n+1} \tag{55}$$

Since in the UV $g(\frac{s}{2}) \approx \frac{1}{\sqrt{s}}$, it is easy to see that the UV behaviour of the overall integrand is $\sim s^{-\frac{3}{2}}$, independently of n. As for the other terms, let us consider in the RHS of (54) the factor that multiplies $\frac{1}{s^{k-l}}$ (for $l \geq 2$). Setting $s = 0$, the summation over $n_1, \ldots, n_{l-1}$ is always convergent, so that the UV behaviour of each term in the summation is given by the factor $\frac{1}{s^{k-l}}$, with $2 \leq l \leq k$. It follows that the most UV divergent term corresponds to $l = 0$, $\sim \frac{1}{s^k}$. Since in (53) this is multiplied by

$$s^n\, g(\frac{s}{2}) \left(-\frac{\partial g(\frac{s}{2})}{g(\frac{s}{2})} \right)^{n-k+1} \tag{56}$$

we see that the UV behaviour of the generic term in (53) is at most as singular as $\sim s^{-\frac{3}{2}}$. *In conclusion the states* $\mathbf{\Psi}_n$*, when contracted with* $\Lambda(\varepsilon)$*, give rise to the same kind of UV singularity* $\sim s^{-\frac{3}{2}}$. Now, for any two such states, say $\mathbf{\Psi}_n$ and $\mathbf{\Psi}_{n'}$, we can form a suitable combination such that the UV singularity cancels. In this way we generate infinite many states, say $\mathbf{\Phi}_n$, which, when contracted with $\Lambda(\varepsilon)$, give rise to UV convergent correlators.

Let us consider next the IR properties ($s \gg 1$). All the correlators contain the factor $e^{-\tilde{\eta} s}$ which renders them IR convergent, but we have learnt that the crucial IR properties (in the limit $\varepsilon \to 0$) are obtained by ignoring this exponential factor. This is in order to guarantee the continuity of the $\varepsilon \to 0$ limit. So, in analyzing the IR properties we will ignore this factor. The first term (55) is very strongly convergent in the IR, because $\partial_s g(\frac{s}{2}) \approx \frac{1}{s^2}$, while $g(\frac{s}{2}) \to 1$. For the remaining terms let us consider in the RHS of (54) the factor that multiplies $\frac{1}{s^{k-l}}$ (for $l \geq 2$). To estimate the IR behaviour it is very important to know the degree difference between the polynomials Q_l and P_l. Above we said that this difference is always nonnegative. In principle it could vanish, but from the example with $n = 2$, see [2], we know that there are cancellations and that in fact the difference in degree is at least 2. If this is so in general, we can conclude that the IR behaviour of the summation in the RHS of (54) with fixed l is $\sim \frac{1}{s^l}$. However, in order to prove such cancellations, one would have to do detailed calculations, which we wish to avoid here. So we will take the pessimistic point of view and assume that, at least for some of the terms, $\mathrm{deg} Q_l = \mathrm{deg} P_l$ (in which case there remains only one angular integration). In this case the IR behaviour of the corresponding term cannot decrease faster than $\sim \frac{1}{s^{l-1}}$. This has to be multiplied by $\sim \frac{1}{s^{k-l}}$ and by the IR behaviour of (56). This means that the least convergent term with fixed k in(54) behaves as $\sim \frac{1}{s^{n-k+1}}$. Since $k \leq n+1$, we see that in the worst hypothesis in the integral (53) there can be linearly divergent terms, before the $e^{-\varepsilon \partial_\varepsilon} \varepsilon$ operator is applied. If this is so the UV converging $\mathbf{\Phi}_n$ states are not good test states. However we can repeat for the IR singularities what we have done for the UV ones. Taking suitable differences of the $\mathbf{\Phi}_n$'s (this requires a two steps process, first for the linear and then for the logarithmic IR singularities, we can create an infinite set of states, $\mathbf{\Omega}_n$, which, when contracted with $\Lambda(\varepsilon)$, yield a finite result and whose $\varepsilon \to 0$ limit is continuous. Upon applying $\Gamma(\varepsilon)$, instead of $\Lambda(\varepsilon)$, they of course vanish. These $\mathbf{\Omega}_n$ are therefore good (and nontrivial) test states. They annihilate $\Gamma(\varepsilon)$.

We remark that in eq.(52) the presence of ε in $H(\phi_u,\varepsilon)$ is not essential, because in estimating the IR behaviour we have not counted the $e^{-\tilde{\eta}s}$ factor. Using $\frac{1}{K+\phi_u}$ everywhere instead of $\frac{1}{K+\phi_u+\varepsilon}$, would lead to the same results. This means that contracting the Ω_n states with Λ_ε leads to finite correlators *with or without* ε. We stress again that *the* $\varepsilon \to 0$ *limit of such correlators is continuous*. This is the real distinctive features of good test states. The property of annihilating $\Gamma(\varepsilon)$, is a consequence thereof. This remark will be used later on.

The $\Omega_n(\phi_u,\varepsilon)$ are however only a first set of good test states. One can envisage a manifold of other such states. Let us briefly describe them, without going into too many details. For instance, let us start again from (52) and replace the first $H(\phi_u,\varepsilon)$ factor with $\frac{1}{K+\phi_u+\varepsilon}uX^{2k}$ (the term $\delta\phi$ can be dropped). In this way we obtain a new state depending on a new integral label k. However replacing X^2 with X^{2k} is too rough an operation, which renders the calculations unwieldy, because it breaks the covariance with respect to the rescaling $z \to \frac{z}{t}$. It is rather easy to remedy by studying the conformal transformation of X^{2k}. The following corrected replacements will do:

$$\begin{aligned} uX^2 &\to u\left(X^2+2(\log u+\gamma)\right) = \phi_u \equiv \phi_u^{(1)} \\ uX^4 &\to u\left(X^4+12(\log u+\gamma)X^2+12(\log u+\gamma)^2\right) \equiv \phi_u^{(2)} \\ &\dots \\ uX^{2k} &\to u\left(\sum_{i=0}^{k}\frac{(2k)!}{(2k-2i)!i!}\left(\log u+\gamma\right)^i X^{2k-2i}\right) \equiv \phi_u^{(k)} \end{aligned} \tag{57}$$

The role of the additional pieces on the RHS is to allow us to reconstruct the derivatives of $g(s)$ in computing the correlators, as was done in [1].

Now let us denote by $\mathbf{\Psi}_n^{(k)}$ the n-th state (52) where $\phi_u - \delta\phi_u$ in the first $H(\phi_u,\varepsilon)$ factor is replaced by $\phi_u^{(k)}$. Contracting it with $\Lambda(\varepsilon)$ it is not hard to see that the term (55) will be replaced by

$$\sim \int_0^\infty ds\, e^{-\tilde{\eta}s}\, s^n\, g(\frac{s}{2})\left(\frac{\partial_s g(\frac{s}{2})}{g(\frac{s}{2})}\right)^{n+k} \tag{58}$$

with analogous generalizations for the other terms. It is evident from (58) that the UV behaviour becomes more singular with respect to (55) while the IR one becomes more convergent. This is a general property of all the terms in the correlator. Thus fixing k we will have a definite UV singularity, the same up to a multiplicative factor for all $\mathbf{\Psi}_n^{(k)}$. Therefore by combining a finite number of them we can eliminate the UV singularity and obtain another infinite set of UV convergent states $\mathbf{\Omega}_n^{(k)}$ for any k ($\mathbf{\Omega}_n^{(1)}$ will coincide with the previously introduced $\mathbf{\Omega}_n$). In general they will be IR convergent (IR subtractions may be necessary for $k=2$ beside $k=1$).

It goes without saying that the previous construction can be further generalized by replacing in (52) more than one X^2 factors with higher powers X^{2k}.

Qualitatively one can say that the correlators discussed so far have the form of an s integral

$$\int_0^\infty ds\, F(s) \tag{59}$$

where the $F(s)$ at the origin behaves as $s^{\frac{k}{2}}$, with integer $k \geq -1$, and F with all possible k's are present. At infinity, excluding the $e^{-\tilde{\eta}s}$, $F(s)$ behaves as $\frac{1}{s^p}$, for any integer $p \geq 2$. In addition, at infinity, we have any possible exponentially decreasing behaviour.

8. The topological vector space of test states

Above we have introduced a countable set of good test states which will be denoted generically by $\mathbf{\Omega}_\alpha$, $\alpha \in \mathbf{A}$ being a multi-index. We recall that in $\mathbf{\Omega}_\alpha$ there is also a dependence on the parameter ε. Such a dependence improves the IR convergence properties. We will include this dependence in the multi-index α. The linear span of these state will be denoted by $\mathfrak{F}$. It is a vector space. The problem now is to define a topology on it. First of all we define

$$\langle \mathbf{\Omega}_\alpha | \mathbf{\Omega}_\beta \rangle \equiv \langle \mathbf{\Omega}_\alpha Bc\partial cB, \Lambda_\varepsilon \rangle \langle \mathbf{\Omega}_\beta Bc\partial cB, \Lambda_\varepsilon \rangle \tag{60}$$

where the RHS is the corresponding correlator. From the analysis of the previous subsection this is a finite number, generically nonvanishing. Whenever a correlator of this kind depends on ε, the limit $\varepsilon \to 0$ exists and is finite. Extending by linearity the definition (60) to all finite combinations of the vectors $\mathbf{\Omega}_\alpha$ we get an inner product. Thus $\mathfrak{F}$ is an inner product space. This inner product is not a scalar product in general. However it is certainly nondegenerate (i.e. there are no elements with vanishing inner product with all the elements of the space). The existence of an inner product does not mean by itself that $\mathfrak{F}$ is a topological vector space.

8.1. Seminorm topology

There are various ways to introduce a topology in an inner product space $\mathcal{V}$, see [23, 24]. We will use seminorms. Let us denote by $x, y, \ldots$ the elements of $\mathcal{V}$, and by (x, y) the inner product. A *seminorm* is a function in $\mathcal{V}$ that satisfies the following axioms

$$\begin{aligned} & p(x) \geq 0 \\ & p(ax) = |a| p(x), \quad a \in \mathbb{C} \\ & p(x+y) \leq p(x) + p(y) \end{aligned} \tag{61}$$

Once we have an (infinite) family p_γ (γ is a generic index) of seminorms we can define a topology τ in the following way: a subset V is open if for any $x \in V$ there is a finite subset $p_{\gamma_1}, p_{\gamma_2}, \ldots, p_{\gamma_n}$ of seminorms and a positive number ϵ, such that, any other element y satisfying $p_{\gamma_j}(x - y) < \epsilon$, for $j = 1, \ldots n$, belongs to V. A topology τ is *locally convex* if the vector space operations are continuous in τ and if a τ-neighborhood of any point x contains a convex neighborhood of the same point.

What we wish is of course a topology strictly related to the inner product. Therefore we introduce the concept of *partial majorant*. A partial majorant of the inner product $(\cdot, \cdot)$ is a topology τ which is locally convex and such that for any $y \in \mathcal{V}$ the function $\varphi_y(x) = (x, y)$ is τ-continuous.

In addition we say that a topology τ is *admissible* if 1) τ is a partial majorant and 2) for any linear τ-continuous functional $\varphi_0(x)$ there is an element $y_0 \in \mathcal{V}$ such that $\varphi_0(x) = (x, y_0)$. That is, all the continuous linear functionals can be expressed as elements of $\mathcal{V}$ via the inner product.

It is easy to prove that in any inner product space the function p_y defined by

$$p_y(x) = |(x, y)| \tag{62}$$

is a seminorm. The corresponding topology is the *weak topology* τ_0. This topology has important properties. The weak topology τ_0 is a partial majorant in $\mathcal{V}$. If the inner product is non-degenerate the space is *separated (Hausdorff)*. Moreover τ_0 is admissible.

When $\mathcal{V}$ is assigned the τ_0 topology, it will be denoted by $\mathcal{V}_w$.

For later use we have to define the concept of bounded set. A subset B is bounded if for any neighborhood V of 0 there is a positive number λ such that $B \subset \lambda V$. In terms of seminorms we can say that B is bounded if all seminorms are bounded by some finite number in B.

Now, let us return to $\mathfrak{F}$ with the inner product $\langle \cdot | \cdot \rangle$ defined via (60). Using it we can define an infinite set of seminorms as above and thereby the weak τ_0 topology. In virtue of the preceding discussion $\mathfrak{F}$ becomes a topological vector space with a separated admissible topology. We can also assume that the second axiom of countability holds for $\mathfrak{F}$. This is due to the fact that, apart from the ε dependence, we can numerate the basis of all possible states Ω_α. As for ε we can discretize it, i.e replace it with a sequence ε_n tending to 0. In this way the index α is replaced by a discrete multi-index ν and we obtain a countable set of seminorms p_ν. The neighborhoods of the origin defined by these seminorms form a countable basis. Finally, $\mathfrak{F}$

with the τ_0 topology is not a normed partial majorant, therefore it is not metrizable.

To stress that $\mathcal{F}$ is equipped with the τ_0 topology we will use the symbol $\mathcal{F}_w$.

This is not yet what we need in order to proceed. The reason is that, since the topology is admissible, any continuous functional can be expressed in terms $\mathcal{F}$, so in the dual there is no room for distributions. The τ_0 topology is 'too weak' to accommodate simultaneously a space of test states and distributions. In $\mathcal{F}$ we need a stronger topology. We say that a topology τ_1 is stronger or finer than τ_2 ($\tau_1 \geq \tau_2$) if any open set in τ_2 is an open set also in τ_1. It is a theorem that if τ is locally convex and stronger than τ_0 it is also a partial majorant, which guarantees continuity of the scalar product also wrt τ. We will shortly introduce on $\mathcal{F}$ the *strong topology.* But to do so we need first to discuss the topology on the dual.

8.2. The dual space and the strong topology

Given a topological vector space $\mathcal{V}$ as above, the dual $\mathcal{V}'$ is the space of linear continuous functionals. Let us denote linear continuous functionals by $x', y', ...$ and their evaluation over a point $x \in \mathcal{V}$ by $x'(x), y'(x), ...$

The *weak topology* over $\mathcal{V}'$ can be defined as follows: a sequence of linear continuous functional x'_n weakly converges to 0, if the numerical sequence $x'_n(x)$ converges to 0 for any $x \in \mathcal{V}$. This topology turns $\mathcal{V}'$ into a locally convex topological vector space.

A subset $B' \in \mathcal{V}'$ is (weakly) bounded if for any neighborhood U'_ϵ of 0 in $\mathcal{V}'$ there exist a positive number λ such that $\lambda B' \subset U'_\epsilon$.

The space $\mathcal{V}'$ with the weak topology will be denoted $\mathcal{V}'_w$.

We can immediately transfer these concepts to the space $\mathcal{F}'$ of linear continuous functionals over $\mathcal{F}$, which is therefore itself a convex topological vector space. The space $\mathcal{F}'$ with the weak topology will be denoted by $\mathcal{F}'_w$.

Using the weak topology on $\mathcal{V}'$ we can now define the *strong topology* on $\mathcal{V}$. The latter is defined as the uniform convergence topology on all weakly bounded subsets of $\mathcal{V}'$. This means that a sequence x_n converges to 0 in $\mathcal{V}$ if the numerical functions $x'(x_n)$ converge to zero uniformly for x' in any bounded subset B of $\mathcal{V}'$.

We recall that when $\mathcal{V}$ is assigned the weak τ_0 topology, for any continuous functional $x' \in \mathcal{V}'$ we have $x'(x) = (x, y)$ for some $y \in \mathcal{V}$. This is generically not true for the dual of $\mathcal{V}$ when $\mathcal{V}$ is equipped with the strong topology. The dual of $\mathcal{V}_s$ is generally larger than $\mathcal{V}'$. The dual of $\mathcal{V}_s$ will be denoted by $\mathcal{V}'_s$. It can itself be equipped with a *strong topology* as follows: a neighborhood V'_ϵ of 0 in $\mathcal{V}'_s$ is defined by

$$V'_\epsilon = \left\{x' \in \mathcal{V}' \middle| \sup_{x\in B}|x'(x)| < \epsilon\right\} \qquad (63)$$

for any ϵ and any bounded set $B \subset \mathcal{V}_s$. $\mathcal{V}'_s$ equipped with the strong topology will be denoted also as $\mathcal{V}'_{ss}$.

We can immediately transfer these concepts to the space $\mathfrak{F}$ and its duals. The space $\mathfrak{F}'$ with the weak topology will be denoted by $\mathfrak{F}'_w$ and $\mathfrak{F}$ with the strong topology will be denoted by $\mathfrak{F}_s$. The dual of the latter will be denoted with the symbol $\mathfrak{F}'_s$

9. The spaces of test states and the space of generalized states

The space $\mathfrak{F}$ equipped with the strong topology will be our *space of test states*. The dual of the latter, i.e. $\mathfrak{F}'_s$ will be our space of *generalized states* or *distributions*. If we equip the latter with the strong topology it will be denoted by $\mathfrak{F}'_{ss}$.

As in ordinary distribution theory we have to verify that $\mathfrak{F}$ is a rich enough filter that no regular behaviour can escape through it. We first remark that the cardinality of the basis $\mathbf{\Omega}_\alpha$ with fixed ε is the same as the cardinality of the Fock space states $\mathcal{F}$. Now let us compare the situation here with sec. 5.3. To be able to claim that $\mathfrak{F}$ is *rich enough* we must show that a state that annihilates the full $\mathfrak{F}$ can only be 0. To see this let us consider a generic finite linear combination of states $\mathbf{\Omega}_\alpha$, say $\mathbf{\Upsilon}$, and suppose that

$$\langle \mathbf{\Upsilon} | \mathbf{\Omega}_\alpha \rangle = 0, \qquad \forall\, \mathbf{\Omega}_\alpha \in \mathfrak{F} \tag{64}$$

But if such a state $\mathbf{\Upsilon}$ were to exist it would mean that the inner product (60) is degenerate. As far as we can exclude the degeneracy of the inner product we conclude that $\mathfrak{F}$ is a rich enough space of test states.

10. Conclusion

In this paper we have set the stage for a rigorous treatment of the problem raised in section 3. We propose to interpret expressions such as (36) and (37) in the framework of a generalized distribution theory, in which the relevant objects are not ordinary functions but string fields. To this end we have shown that it is possible to introduce a space of test string fields and define a weak and strong topology on it. Then we have defined the dual space of linear functionals together with its strong and weak topology. We claim that the string fields of SFT must to be interpreted as elements of this dual space. This must be the case in particular for $\mathcal{A}_\varepsilon$.

In [2, 4] arguments to support our claim that $\mathcal{A}_\varepsilon$ is zero were put forward. In this paper we have produced more evidence. In particular, as we have noted, the space of test states $\mathfrak{F}$ annihilates $\mathcal{A}_\varepsilon$. This is another element in favor of our thesis. However we cannot yet claim that this is a rigorous proof. For this to be the case we will have to prove that $\mathcal{A}_\varepsilon$ is zero as a limit in the strong dual topology.

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